Toroidal Obstructions with No K_5 Minor

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Abstract

Finding the complete set of forbidden minors for the torus remains an open problem. We determine the complete list of toroidal minor obstructions with no K_5 minor and prove that there are exactly 74, each with between 10 and 14 vertices. Combined with the known classification of toroidal obstructions with no $K_{3,3}$ minor, this determines every toroidal obstruction that excludes one of the forbidden minors for planarity. The proof uses three-vertex cuts to show that, apart from $K_{3,7}$, every three-connected obstruction in this class has two distinct independent triples, each the common neighborhood of at least two degree-three vertices. The classification proceeds by considering the possible intersections of these triples and reduces each case by planarity tests and enumeration of the possible graph forms. From the 74 minor obstructions we also obtain a complete list of 381 topological obstructions for the torus with no K_5 minor.

Keywords. Toroidal graph, graph embedding, forbidden minor, graph genus.

Mathematics Subject Classification (2020). 05C10, 05C83.

1 Introduction

In 1930, Kuratowski proved that a graph can be drawn in the plane without any edge crossings if and only if it contains no subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$ [14]. In 1937, Wagner showed that the same two graphs are exactly the complete set of forbidden minors for the plane [28]. Both graphs can, however, be embedded on the torus. A natural next question is: Which graphs that cannot be embedded in the plane can still be embedded in the torus?

A graph is *toroidal* if it can be embedded on the torus. A *minor obstruction*, or forbidden minor, for a surface is a graph that cannot be embedded on that surface while every proper minor of it can. A toroidal obstruction is a minor obstruction for the torus. Robertson and Seymour proved that every fixed surface has only finitely many minor obstructions [26].[1] Outside of planar graphs, the complete set of forbidden minors is also known for the projective plane. The work of Glover, Huneke and Wang, together with Archdeacon's completeness proof, gives a list of 35 forbidden minors [9, 1]. Despite the results of Robertson and Seymour, outside of the plane and projective plane there is little known about the complete set of forbidden minors for other surfaces.

For the torus, exhaustive searches by computer first reached ten vertices [24], then eleven [5] and twelve [23], and the number of known minor obstructions rose to 2,249 and then to 16,682 [5, 23]. Myrvold and Woodcock collected 17,535 minor obstructions [23], but the collection is believed to be incomplete. Among graphs with no K_5 minor, their collection turns out to be one graph short. The missing graph is $\mathfrak{T}_{74}$ in the list of Theorem 1.1 below [Supp. O4]. The supplemental data also includes thirteen further toroidal minor obstructions, each containing a K_5 minor, that are not in their collection [Supp. N1].

Battle, Harary, Kodama and Youngs proved that the orientable genus of a graph is additive over its blocks [2]. Thus, there is exactly one disconnected toroidal obstruction with no K_5 minor and exactly one with a cut vertex, namely $\mathfrak{T}_{25}$ and $\mathfrak{T}_8$. Mohar and Škoda classified all 68 toroidal obstructions of vertex-connectivity two [18, Theorem 8.10 and Corollary 8.11]. A plausible approach to completing the classification of the forbidden toroidal minors could be a piecemeal approach as suggested in [23].

Gagarin and Kocay studied the structure of graphs containing a K_5 subdivision but no $K_{3,3}$ subdivision [7] and further, Gagarin, Myrvold and Chambers proved that there are exactly four toroidal minor obstructions with no $K_{3,3}$ minor, and eleven topological obstructions [8]. By a theorem of Hall, K_5 is the only three-connected nonplanar graph with no $K_{3,3}$ minor [12], and K_5 is toroidal, so each of the four minor obstructions has connectivity at most

[1] Seymour's later bound on the size of such a graph is $2^{2^{15^9}}$ vertices for the torus [27], while the largest known toroidal minor obstruction has 18 vertices.

two. Since $K_{3,3}$ is cubic, excluding a $K_{3,3}$ minor is equivalent to excluding a $K_{3,3}$ subdivision [8, Section 1]. In this paper, we classify toroidal obstructions with no K_5 minor, which instead form a proper subclass of those with no K_5 subdivision. Wagner showed that every graph with no K_5 minor is a subgraph of one obtained by gluing planar graphs and copies of the eight-vertex Wagner graph V_8 along cliques of order at most three [28]. Four-colorings of the pieces can be made to agree on a shared clique by permuting the colors. Thus the four color theorem is equivalent to the assertion that every graph with no K_5 minor is four-colorable. Halin's theorem, which we use below, comes from his classification of finite graphs according to Hadwiger and Wagner [10, 11].

Theorem 1.1. *A finite simple graph has no K_5 minor, is nontoroidal and has every proper minor toroidal if and only if it is isomorphic to one of $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$, given in Table 2 and* [Supp. O1].

Combining this with the result of Gagarin, Myrvold and Chambers determines every toroidal obstruction that excludes at least one of the two forbidden minors for planarity. Any obstruction outside these two classifications must therefore contain both a K_5 minor and a $K_{3,3}$ minor.

As noted, Mohar and Škoda's results classify the toroidal obstructions of connectivity two, including those without a K_5 minor. Additionally, any nonplanar four-connected graph contains a K_5 minor by Wagner's theorem [28]. Hence, it remains only to consider the three-connected obstructions which is the goal of this paper. For the three-connected case, we use Halin's theorem [10, 11] to obtain three-vertex cuts and study the components they separate. Apart from $K_{3,7}$, we show every three-connected obstruction has two distinct independent triples, each the common neighborhood of at least two degree-three vertices (Theorem 3.6). We call such a triple a twin triple. Section 3 proves this theorem and develops lemmas on bridges used throughout. The classification then splits according to whether the triples have no common vertex, two common vertices or one, treated in Sections 4, 5 and 6. Section 7 assembles the proof of Theorem 1.1. Appendix A lists the supplemental data items and describes how each is checked. Table 1 lists the cases and the graphs obtained in each.

Each of Sections 4, 5 and 6 follows the same steps. We first find the faces of a subgraph containing the two triples and four of the degree-three vertices, and reduce the toroidality of G to planarity tests on the bridges of this subgraph. By minimality, no bridge can then be reduced while its nonplanar tests stay nonplanar (Lemma 3.7). We finally retain a $K_{3,3}$ subdivision in a nonplanar test, which bounds the size of the bridge, and enumerate the possible forms (Corollary 3.19). The three sections are independent of one another, with a few small exceptions.

A *topological obstruction* for the torus is a nontoroidal graph of minimum degree at least three in which every edge deletion is toroidal. In Section 8 we use Theorem 1.1 to reduce the topological obstructions with no K_5 minor to a search by repeated vertex splitting starting from $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$, and find that there are exactly 381 (Theorem 8.1).

2 Preliminaries and basic reductions

For the convenience of the reader we provide the following preliminary results and definitions. When possible, we follow the terminology of *Graphs on Surfaces* by Mohar and Thomassen [21]. Graphs are finite, simple, and undirected, unless stated otherwise. We use $\cup$ for graph unions, $+$ for adding specified vertices or edges, and $K_n[X]$ for the complete graph on a set X of n vertices. We write $N_G(X)$ for the vertices outside X with a neighbor in X. By Youngs's theorem [30], every embedding of a connected nonplanar graph on the torus is cellular. In this paper, an *obstruction* will mean a toroidal minor obstruction with no K_5 minor. Unless stated otherwise, G denotes a three-connected obstruction. By minor-minimality, any nontoroidal minor of G is isomorphic to G.

For a subgraph $K \subseteq G$, a *K-bridge* B is either a component of $G - V(K)$ together with its neighbors in K and all edges joining that component to K, or an edge of $E(G) \setminus E(K)$ with both ends in K. Its vertices in K form its *attachment set* A, and $\mathrm{Int}(B) = V(B) \setminus A$ is its *interior* vertex set. When K has no edges, we call its bridges *bridges on $V(K)$*. A subscript in B_X specifies the defining subgraph or vertex set when needed. Otherwise we omit it, and state the ambient graph whenever it changes. Auxiliary graphs are denoted by H, with subscripts when needed. The *exterior* of a K-bridge B is obtained from G by deleting the interior vertices of B and all edges of B. Every edge of G at an interior vertex of B belongs to B, and adding edges between attachments creates no new edge there. Thus, for any vertex set X and any set E of edges between attachments, a set of interior vertices of B that forms a component of one of $G - X$, $B - X$ and $(B + E) - X$ forms a component of all three.

Let K be a connected graph embedded in an orientable surface Σ, and let $\mathcal{N} \subseteq \Sigma$ be a closed regular neighborhood of K. Then $\mathcal{N}$ is a compact connected orientable surface with boundary, $K \subseteq \operatorname{int}\mathcal{N}$. The genus of $\mathcal{N}$ is that of the closed surface obtained by gluing a disk to each boundary component of $\mathcal{N}$.

Lemma 2.1. *Let $\mathcal{N}$ be a closed regular neighborhood of a connected graph embedded on the torus, and let M be the closure of its complement. If $\mathcal{N}$ has genus one, every component of M is a disk. If $\mathcal{N}$ has genus zero, M consists of disks and exactly one annulus or one torus with one open disk removed.*

Case	Settled by	i with $G \cong \mathfrak{T}_i$
Not two-connected	Section 7	8, 25
Connectivity two	Mohar and Škoda [18]	3, 6, 12, 13, 27, 28, 31, 36, 42, 44, 46, 47, 50, 60, 62, 64, 65, 71
Three-connected, $K_{3,7}$	Theorem 3.6	1
Two disjoint twin triples	Theorem 4.1	15 graphs
The interior of one path is joined to two vertices of one triple	Lemma 4.4	4
The attachments of a bridge lie on no common face of Ψ_1	Lemma 4.5	5
All three paths are edges	Theorem 4.22	5, 48, 58, 68, 69
Some path is not an edge and every bridge with four attachments has one interior vertex	Proposition 4.24	32, 38, 43, 45, 48, 61, 66, 67
A bridge with four attachments and at least two interior vertices, of type (c)	Proposition 4.25	21, 63
Of type (a)	Proposition 4.28	none
Of type (b)	Proposition 4.30	66, or an earlier row
Two twin triples with two common vertices	Theorem 5.1	24 graphs
A bridge with four attachments and four vertices with the same neighborhood	Lemma 5.2	2
A bridge B with four attachments and $(n_c, n_d) = (3, 3)$	Lemma 5.6	none
B nonplanar with $(n_c, n_d) = (2, 2)$, or $(n_c, n_d) = (3, 2)$	Proposition 5.9	11, 17, 18, 19, 22, 24, 26, 30, 34, 39, 52, 57, 74, and 48, 66 again
B planar with $(n_c, n_d) = (2, 2)$ and a cut $\{a, b, w\}$	Proposition 5.16	56, 72, 73
A cut $\{a, u, v\}$	Proposition 5.24	40, 51, 53, 56, 72, 73
A cut $\{a, c, w\}$	Proposition 5.27	none
No bridge with four attachments and all triangle completions planar	Lemma 5.28	9, 10, 16
No bridge with four attachments and some triangle completion nonplanar	Proposition 5.33	15
Two twin triples with one common vertex	Theorem 6.1	14 graphs
A bridge with five attachments	Proposition 6.8	20, 23
A bridge with four attachments, not including a	Proposition 6.13	29, 54, 55
A bridge with four attachments, including a	Proposition 6.16	14, 37, 41, 59
Every bridge has three attachments	Proposition 6.25	7, 33, 35, 49, 70

Table 1: The cases of the classification and the graphs obtained in each.

Proof. Let g, b be the genus and number of boundary components of $\mathcal{N}$, and let $M_1, \ldots, M_k$ be the components of M, with genera h_i and boundary counts b_i. Every boundary circle of $\mathcal{N}$ borders one of the M_i, so $\sum_i b_i = b$. Since each $b_i \geq 1$, we have $b \geq k$. The Euler characteristic of a connected orientable surface of genus h with b boundary components is $2 - 2h - b$. Gluing along the boundary circles gives the torus, so additivity of Euler characteristic gives

$$0 = \chi(\mathcal{N}) + \sum_{i=1}^{k} \chi(M_i) = (2 - 2g - b) + \left(2k - 2\sum_{i=1}^{k} h_i - b\right).$$

Thus $b = 1 - g + k - \sum_i h_i$. If $g = 1$, then $b = k - \sum_i h_i$. If any $h_i > 0$, then $b < k$, a contradiction. Thus every $h_i = 0$ and $b = k$, so every $b_i = 1$. Every component of M is therefore a disk. If $g = 0$, then $b = 1 + k - \sum_i h_i$. Since $b \geq k$, we have $\sum_i h_i \leq 1$. If one $h_i = 1$, then $b = k$ and every $b_i = 1$; we obtain one torus with an open disk removed, and all other components are disks. Otherwise every $h_i = 0$ and $b = k + 1$, so exactly one $b_i = 2$ and all the others are one, and we obtain one annulus, and all other components are disks. □

2.1 Extending an embedding

Let H be a graph, $K \subseteq H$ a connected subgraph, and Π an embedding of K on the torus. We say that Π extends to an embedding of H if H has an embedding that agrees with Π on every vertex and edge of K. *Assigning* a K-bridge B to a face F of Π means choosing that face as the region in which B is to be drawn outside of its attachments. Every attachment of B must occur on a boundary walk of F.

We refer to the cyclic list of attachment appearances along a facial walk as its *attachment sequence*, written for example as *abcad*. A vertical bar separates boundary components, as in $ac \mid ad$. In an auxiliary drawing with separated boundary appearances, a_1, a_2 denote appearances of the same original vertex a.

Suppose Π extends to an embedding of H. For each K-bridge B, remove only the points representing its attachments from its drawing. The remaining set is connected and disjoint from $\Pi(K)$, so it lies in one face of Π. Thus, deciding whether Π extends to an embedding of H amounts to deciding whether all its bridges can be assigned to faces and drawn all together at once in their chosen faces without crossings.

The next lemma gives the test that we use most often for a single bridge in a disk face.

Lemma 2.2. *Let B be a bridge with nonempty connected interior. Suppose its attachments appear around the boundary of a disk face F in one of the cyclic sequences*

$$a, \quad ab, \quad abc, \quad abac, \quad abcd, \quad abcad, \quad abacad, \quad abcdcb,$$

up to relabeling and reversal. Then B can be drawn in F if and only if the graph obtained from B by joining each pair of distinct consecutive attachments in the sequence is planar.

Proof. For the sequence a, a plane drawing of B with a on the outer face can be placed in F, and a drawing in F is planar. For the other sequences, let H_0 be the graph consisting of the attachments and the edges joining consecutive attachments. It has a plane embedding with the listed facial walk, and every facial walk containing all its vertices has the same cyclic order, up to reversal. If $B \cup H_0$ is planar, the connected interior of B lies in one face of H_0. Its boundary contains every attachment, so its boundary walk gives the required drawing in F. Conversely, inserting a drawing of B into the corresponding face of H_0 gives a planar drawing of $B \cup H_0$. □

Figure 1 shows the sequence *abcad*.

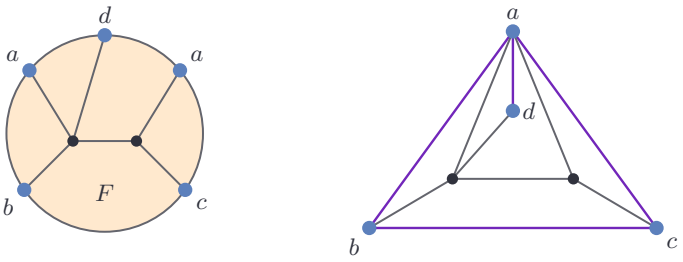


Figure 1: Left, a bridge in a disk face F with attachment sequence *abcad*. Right, the same bridge with the consecutive attachments joined.

Let $\mathcal{N}$ be a closed regular neighborhood of $\Pi(K)$, formed by taking small closed disks around the vertices and connecting them by narrow closed regions that follow the edges within the surface.

For a graph H containing a cycle C, let $\mathrm{Aux}(H, C)$ be H plus a new vertex adjacent precisely to the vertices of C. We consider the cases where $\mathcal{N}$ has genus one and genus zero separately. Suppose first that $\mathcal{N}$ has genus one. By Lemma 2.1, every face of Π is a disk. The following construction adapts the split auxiliary graph of Mohar and Thomassen [21, p. 191] to record the specific chosen boundary appearances.

For each disk face F chosen for at least one bridge B, we construct graphs H_F, C_F and H_F^+ as follows. For each edge of a bridge B assigned to F, and each endpoint v of that edge belonging to K, choose an appearance of v on the boundary walk of F. List the distinct chosen appearances in cyclic order and represent them by distinct new vertices $x_1, \ldots, x_k$. Define $V(H_F)$ to be $\{x_1, \ldots, x_k\}$ together with all the interior vertices of the bridges assigned to F. Every edge of a bridge assigned to F defines one edge of H_F. An edge uw between interior vertices is kept. An edge from an interior vertex u to an attachment v becomes ux_i, where x_i represents the appearance chosen at v. An edge between attachments becomes x_ix_j, where x_i and x_j represent the appearances chosen for that edge at its two endpoints. Set $m = \max\{k, 3\}$. If $k < 3$, introduce new vertices $x_{k+1}, \ldots, x_m$, distinct from all vertices of H_F. Define C_F to be the cycle $x_1x_2 \cdots x_mx_1$. The vertices $x_1, \ldots, x_k$ follow the chosen cyclic boundary order. Every edge of C_F is a new edge. Set $H_F^+ = \mathrm{Aux}(H_F \cup C_F, C_F)$, retaining any parallel edges.

Lemma 2.3 (Adding bridges in a face homeomorphic to a disk)**.** *With the notation above, the bridges assigned to a disk face F can be added to Π inside F, through their chosen boundary appearances, if and only if H_F^+ is planar.*

Proof. By Mohar's disk criterion [17, Section 3, first paragraph], applied to $H_F \cup C_F$ with boundary cycle C_F, the bridges assigned to a disk face F can be added to Π inside F through their chosen boundary appearances if and only if H_F^+ is planar. □

Consequently, when $\mathcal{N}$ has genus one, Π extends to an embedding of H if and only if each K-bridge can be assigned to a face of Π and its boundary appearances chosen so that every graph H_F^+ is planar.

Now suppose that $\mathcal{N}$ has genus zero. By Lemma 2.1, the faces of Π are disks together with exactly one annulus or exactly one torus with one open disk removed. Suppose the case where some face is a torus with one open disk removed has been excluded, so that there is one annular face F and all other faces are disks. Represent F by the interior of a closed annulus Σ_0 whose boundary components Γ_1, Γ_2 trace the two boundary walks of F. Assign a collection of K-bridges in H to F. For each edge of an assigned bridge and each endpoint $v \in V(K)$, choose an appearance of v on one of these boundary walks, with at most two distinct chosen appearances on each walk. Define H_{ann} to have one new vertex for each distinct chosen appearance, together with all the interior vertices of the assigned bridges. Define its edges by the same endpoint rules as H_F. Appearances of the same attachment on different boundary walks are represented by distinct vertices of H_{ann}. For $i = 1, 2$, let $A_i \subseteq V(H_{\text{ann}})$ consist of the vertices representing chosen appearances on the boundary walk traced by Γ_i. For each $v \in A_i$, let $p_v \in \Gamma_i$ be the boundary point representing that appearance. To define H^+_{ann}, label the vertices of A_1 by a, b and those of A_2 by c, d, introducing a new vertex for each unused label. The new vertices are distinct from one another and from every vertex of H_{ann}. Then H^+_{ann} has vertex set $V(H_{\text{ann}}) \cup \{a, b, c, d\}$, the edges of H_{ann}, and two new edges ab, cd.

Lemma 2.4 (Adding bridges in a face homeomorphic to an annulus)**.** *With the notation above, H_{ann} has an embedding in Σ_0 placing each $v \in A_1 \cup A_2$ at p_v, with all other vertices and all edge interiors in $\operatorname{int}\Sigma_0$, if and only if H^+_{ann} is planar.*

Proof. If H_{ann} has the required annulus embedding, place any auxiliary vertices on the corresponding boundary components, cap the boundary components with disks, and add the edges ab, cd through the respective disks. This gives a sphere embedding of H^+_{ann}. Conversely, suppose that H^+_{ann} has a sphere embedding. Take this embedding and trace two disjoint closed disks around ab and cd, meeting the drawing precisely in these respective edges, with the endpoints on the corresponding disk boundaries. Remove the disk interiors, the extra edges, and any auxiliary vertices that were introduced, and we have an annulus embedding of H_{ann}. There are at most two marked vertices on each boundary component, so their cyclic order has no further restriction. A homeomorphism from this annulus to Σ_0 can therefore place every vertex of $A_1 \cup A_2$ at its prescribed point. □

Assign every K-bridge to the annular face or to a disk face, and choose its boundary appearances. If these choices satisfy the hypothesis of Lemma 2.4, the bridges can be drawn together in their assigned faces through those appearances if and only if H^+_{ann} and every H^+_F for a chosen disk face are planar. Drawings in distinct faces have disjoint interiors, so together with $\Pi(K)$ they give an embedding of H.

2.2 Graphs with no K_5 minor

The *Wagner graph* V_8 is obtained from an 8-cycle by joining each pair of opposite vertices. It is triangle-free and toroidal [Supp. W1]. A *three-cut* is a set of three vertices whose deletion disconnects the graph. Halin's theorem implies that every three-connected nonplanar graph with no K_5 minor is isomorphic to V_8 or has a three-cut whose deletion leaves at least three components [10, 11], as stated in [13, Theorem 7]. In particular, such a graph has a three-cut of this kind whenever it contains a triangle. Since V_8 is toroidal, every three-connected obstruction has connectivity exactly three.

2.3 Planarity lemmas

The next lemma gives conditions on individual bridges that determine whether they can be drawn together.

Lemma 2.5. *Let H be a graph and let $A \subseteq V(H)$ have three vertices. Consider the bridges on A.*

(i) *Three bridges adjacent to all of A make H nonplanar.*

(ii) *If exactly two bridges B_1, B_2 are adjacent to all of A, then H is planar if and only if $B_1 \cup K_3[A]$ and $B_2 \cup K_3[A]$ are planar, every other bridge B with two attachments α, β has $B + \alpha\beta$ planar, and every bridge with at most one attachment is planar.*

(iii) *If exactly one bridge B_1 is adjacent to all of A, let E^* consist of the edges $\alpha\beta$ for which another bridge has attachment set $\{\alpha, \beta\}$. Then H is planar if and only if $B_1 + E^*$ is planar and the other bridges satisfy the last two conditions in (ii).*

Proof. For (i), contracting the interiors of three bridges adjacent to all of A gives a $K_{3,3}$ minor. Hence H is not planar.

For the forward direction of (ii), suppose that H is planar. With $\{i, j\} = \{1, 2\}$, delete all bridges besides B_i, B_j, contract the interior of B_i to w, and then replace the three edges at w with a triangle on A. This gives a plane drawing of $B_j \cup K_3[A]$ for both choices of j.

If a bridge with at most one attachment were nonplanar, then H would be nonplanar. Such planar bridges can be removed and reinserted without affecting planarity. If a bridge B with two attachments α, β has $B + \alpha\beta$ nonplanar, then an α–β path through the interior of B_1 shows that H has $B + \alpha\beta$ as a minor and is also nonplanar. This applies to (iii) as well.

For (iii), suppose the other bridges satisfy the stated conditions. Remove the planar bridges with at most one attachment. By the planar bridge replacement theorem [23, Theorem 3.1], each nontrivial two-attachment bridge can be replaced by the edge joining its attachments without changing planarity. The resulting graph is $B_1 + E^*$. Since these replacements and the removal of the other planar bridges do not change planarity, this proves both directions of (iii).

For the reverse direction of (ii), suppose all the stated conditions hold. Since $B_1 \cup K_3[A]$ and $B_2 \cup K_3[A]$ are planar, Lemma 2.2 gives disk drawings of B_1, B_2 with A on their boundaries. Gluing them along their boundary triangles gives a sphere drawing of $B_1 \cup B_2 \cup K_3[A]$. Delete the triangle edges not arising from two-attachment bridges and reverse the replacements above. Reinsert the planar bridges with at most one attachment. This gives a plane drawing of H. □

For a subdivision of $K_{3,3}$, the six vertices of degree three are its *corner vertices*, the nine paths between corner vertices with no corner vertex in their interiors are its *branch paths*, and the vertices of degree two, which lie in the interiors of the branch paths, are its *interior vertices*.

Lemma 2.6. *Let H be a $K_{3,3}$ subdivision containing the edge xy, let P be its branch path through xy, with ends s, t, and let C be the cycle formed by the four branch paths of H avoiding s and t. For a vertex $z \notin \{x, y\}$, possibly outside H, the graph $H - xy + xz + yz$ is nonplanar if and only if $z \notin V(C)$.*

Proof. Use the corner sets $\{s, a_1, a_2\}$ and $\{t, b_1, b_2\}$, and write S_{uv} for the original branch path joining corners u, v. After interchanging x, y if necessary, $P = S_{st}$ passes through $s, \dots, x, y, \dots, t$ in this order.

Suppose $z \notin V(C)$. If $z \notin V(H)$, replacing xy on P by the path xzy gives a $K_{3,3}$ subdivision. The left side of Figure 2 shows this replacement, with the highlighted paths $W_s = P[s, x] \cup xz$ and $W_t = P[y, t]$.

Otherwise, $z \in V(H)$ lies on one of the five branch paths $S_{a_1t}, S_{a_2t}, S_{sb_1}, S_{sb_2}, P$ outside C. If $z \in V(P[s, x])$, replace P by $P[s, z] \cup zy \cup P[y, t]$. If $z \in V(P[y, t])$, use $P[s, x] \cup xz \cup P[z, t]$. Either replacement gives a $K_{3,3}$ subdivision, including when $z = s$ or $z = t$.

The other four cases are symmetric, up to interchanging a_1, a_2, b_1, b_2, or the two corner sets. Without loss of generality, suppose z is an internal vertex of S_{a_1t}. Then there is a $K_{3,3}$ subdivision on corner sets $\{s, a_1, a_2\}$ and $\{z, b_1, b_2\}$. To show this, we must only demonstrate the three branch paths ending at z, which we can take as

$$P[s, x] \cup xz, \qquad S_{a_1t}[a_1, z], \qquad S_{a_2t} \cup S_{a_1t}[t, z].$$

These paths are internally disjoint from one another and from the six unchanged branch paths to b_1, b_2. Suppose $z \in V(C)$. Draw H with P removed except for its ends, with C as a circle, s inside and t outside. The two paths from s cut the inside into two faces whose boundary arcs cover C, and the two paths from t do the same outside. So z lies on an inside face with s and on an outside face with t. Drawing $P[s, x]$ and xz in the first and $P[y, t]$ and yz in the second gives a plane drawing of $H - xy + xz + yz$. □

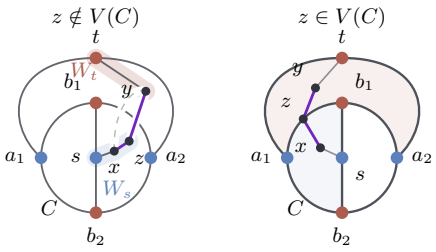


Figure 2: Lemma 2.6. Left, $z \notin V(H)$. Right, a plane drawing of $H - xy + xz + yz$ when $z \in V(C)$.

Lemma 2.7. *If H is connected and planar, $ab \in E(H)$, $d \notin \{a, b\}$, and $H + ad, H + bd$ are nonplanar, then $(H + ad + bd)/ab$ is nonplanar.*

Proof. We prove the contrapositive, so suppose $H_1 = (H + ad + bd)/ab$ is planar. Take a plane drawing of H and form the multigraph H_2 by contracting ab to a vertex r, retaining any parallel edges. The edges at r coming from a are consecutive around r, and so are those coming from b. Merging the parallel edges of $H_2 + rd$ gives H_1, so

drawing each extra copy right beside its edge in a plane drawing of H_1 shows that $H_2 + rd$ is planar. To form a graph without parallel edges, subdivide every edge of H_2 once, giving H_2^*. It is connected because H is, and $H_2^* + rd$ is planar because $H_2 + rd$ is. Placing the new vertices on the curves of H_2 gives a plane drawing of H_2^* with the same order of edges around r.

If r and d do not share a face of H_2^*, we apply [6, Lemma 6] to this drawing, with H_2^*, r, d in the roles of G, x, y. This gives edge sets $E_1 \subsetneq \cdots \subsetneq E_t$, each meeting its complement in one or two vertices and with every edge at r in E_1, such that flipping them in turn makes r and d share a face. At a two-vertex separation, reflect the chosen part while fixing the shared vertices. At a one-vertex separation, choose a face of the part at the shared vertex as its outer face and insert it into a face of the rest at that vertex, reflecting if needed. Since every E_i contains all edges at r, the flips preserve or reverse their cyclic order and keep the edges from each of a, b consecutive. Draw rd in the resulting common face and replace each subdivided edge by a single edge. Assign the new edge to a or b according to the original endpoint of an edge immediately beside it at r. The two groups remain consecutive around r, so split r back into a, b joined by ab, restoring every original endpoint. This gives a plane drawing of $H + ad$ or $H + bd$. □

For a cycle C, a *tripod* consists of a subdivision of $K_{2,3}$ and three disjoint attachment paths, possibly trivial, joining its three branch vertices of degree two to C. Each path meets the subdivision only at its first vertex, and the tripod meets C only at the last vertices of these paths, its *attachment triple* [17].

Theorem 2.8 ([17, Theorem 3.1])**.** *Let C be a cycle in a graph H. If H has no embedding in a closed disk with C on its boundary, then $H - E(C)$ contains* (a) *two disjoint crossing paths whose ends a, c and b, d are distinct vertices of C in the cyclic order a, b, c, d and whose other vertices avoid C,* (b) *a tripod, or* (c) *a Kuratowski subgraph in a three-connected component of $\mathrm{Aux}(H, C)$ other than that containing C.*

Lemma 2.9. *Let B consist of a nonempty connected graph D, four distinct vertices a, b, c, d outside D, and edges joining each of these vertices to D. Suppose B is drawn in a disk with a, b, c, d on the boundary in this cyclic order and D in the interior. Then either B can be redrawn with boundary order a, b, d, c, keeping every vertex of degree one on the original outer face on the outer face, or B contains two vertex-disjoint paths, one from a to d and one from b to c, whose internal vertices lie in D.*

Proof. Consider the drawing as a plane drawing with outer facial walk W. Each of a, b, c, d occurs once on W. Consider the subwalks from d to a and from b to c that avoid the other two vertices. If they are vertex-disjoint, they contain the required paths. Otherwise, choose a common vertex $x \in V(D)$ and an appearance of x on each subwalk. An arc in the outer face between these appearances closes at x to form a simple closed curve meeting the drawing only at x and separating a, b from c, d. Draw the two parts in disks meeting only at x and reflect the part containing c, d. This gives boundary order a, b, d, c. The two appearances of x split W into the outer walks of the two parts. After the reflection, the outer walk is formed by these two walks joined at x, so every vertex of W, in particular every vertex of degree one on it, stays on the outer face. □

3 Two distinct common neighborhoods

We call distinct vertices of degree three with the same neighborhood *twins*. Two twins, their three common neighbors and the six edges between them form a copy of $K_{2,3}$, which we call their *twin subgraph*. The main result of this section, Theorem 3.6, shows that apart from $K_{3,7}$, G has two distinct independent triples, each the neighborhood of at least two twins.

We find twins through three-cuts, using Halin's theorem. For a three-cut X and a component D of $G - X$, write $B_X = G[V(D) \cup X] - E(G[X])$ for its bridge in G and call $H_D = B_X \cup K_3[X]$ the *triangle completion* of D. When $G - X$ has at least three components, H_D is a three-connected proper minor of G by Lemma 3.1. If H_D is planar, then D is a single vertex with neighborhood X by Lemmas 3.1 and 3.2, and two such vertices are twins.

Lemma 3.1. *We have $N(D) = X$, and H_D is three-connected. If $G - X$ has at least three components, then H_D is a proper minor of G, hence K_5-minor-free and toroidal.*

Proof. Three-connectivity gives $N(D) = X$. Since X is a clique in H_D, any vertex cut of the completion with at most two vertices would also separate G. If two other components besides D exist, contract them onto distinct vertices of X to create a triangle, then delete any vertices outside $V(D) \cup X$. □

Lemma 3.2. *If H_D is planar, then D is a single vertex.*

Proof. Since H_D is planar, Lemma 2.2 gives a disk drawing of B_X meeting the boundary precisely in X. If $|V(D)| > 1$, contracting D to a vertex v gives a toroidal proper minor. Replacing v and its three incident edges by that drawing inside a thin closed disk meeting the rest of the embedding precisely in X, using the Jordan–Schoenflies theorem gives a toroidal embedding of G, a contradiction. □

Lemma 3.3. *Let B be a planar bridge of G with attachment set A, and let X be a set of three vertices of B. Every component of $B - X$ disjoint from A is a component of $G - X$ with neighborhood X. There are at most two such components, and if there are two, both are single vertices.*

Proof. Such a component D consists of interior vertices of B. Every edge of G at an interior vertex of B belongs to B, so every neighbor of D in G outside D lies in X. Thus D is a component of $G - X$, and three-connectivity gives $N_G(D) = X$. Its bridge on X in B is therefore adjacent to all of X, so Lemma 2.5(i) allows at most two such components. If there are two, no other bridge on X in B is adjacent to all of X, so part (ii) makes both triangle completions planar, and Lemma 3.2 makes both components single vertices. □

Lemma 3.4. *Let p be a vertex of degree three in an obstruction. Then (a) the neighbors of p are independent, and (b) if the obstruction is three-connected and p, r have twins with $N(p) \neq N(r)$, then p and r are not adjacent.*

Proof. (a) follows from [19, Lemma 24]. (b) Let q and z be the twins of p and r, and suppose pr is an edge. Then $N(p) = N(q) = \{r, z, b\}$ and $N(r) = N(z) = \{p, q, d\}$ for some b, d. The vertices p, q, r, z are distinct as $N(p) \neq N(r)$, and b, d lie outside them as twins are not adjacent. Thus deleting b, d separates $\{p, q, r, z\}$ from the other vertices, which exist since obstructions have more than seven vertices. This contradicts three-connectivity. □

The next lemma finds two pairs of twins inside D.

Lemma 3.5. *Let $Z \subseteq V(G)$. Suppose G has a three-cut X such that $G - X$ has at least three components, one of which, D, avoids Z and has H_D nonplanar, and choose X and D with $|V(D)|$ as small as possible. Then G has a three-cut X' such that the components of $G - X'$ are*

(a) *at least two single vertices in D, each with neighborhood X', and*

(b) *one further component D', which contains $X - X'$ and every component of $G - X$ other than D, and has $H_{D'}$ nonplanar.*

Proof. By Lemma 3.1, H_D is three-connected and has no K_5 minor. It is nonplanar and has a triangle on X, so Halin's theorem gives a three-cut X' of H_D leaving at least three components. Since $H_D - X = D$ is connected, $X' \neq X$. Also, as $X' \subseteq V(H_D) = V(D) \cup X$ and $|X'| = |X| = 3$, X' meets D and $X - X'$ is nonempty. Its vertices are a clique in H_D and thus lie in one component D_0 of $H_D - X'$. Every other component D_i lies in $D - X'$, and there are at least two. Each D_i consists of interior vertices of the bridge B_X, and H_D is obtained from B_X by adding edges between attachments. Thus D_i is also a component of $G - X'$.

Each D_i avoids Z since D does, and $|V(D_i)| < |V(D)|$, since $D_i \subseteq D - X'$ and we showed X' meets D. The vertices $X - X'$ lie neither in X' nor in any D_i, so there is a further component of $G - X'$. There are at least two D_i, so X' is a three-cut with at least three components. If some H_{D_i} were nonplanar, this would contradict the minimal choice of X, D. Hence they are planar, and Lemma 3.2 implies each D_i is a single vertex, with neighborhood X' by Lemma 3.1.

Let D' be the subgraph of $G - X'$ induced by all vertices outside the D_i. These are the vertices of D_0 and of every component of $G - X$ other than D. Each component of $G - X$ other than D has neighborhood X by Lemma 3.1, so it is joined to $X - X' \subseteq V(D_0)$. Any added triangle edge used in D_0 can be replaced by a path through one of these components. Thus D' is connected in $G - X'$. Since the D_i are components of $G - X'$, no edge joins D' to any D_i, so D' is also a component. Finally, D' contains $X - X'$ and at least two components of $G - X$, so $|V(D')| > 1$. Lemma 3.2 therefore implies that $H_{D'}$ is nonplanar. □

Theorem 3.6. *If $G \not\cong K_{3,7}$, then G has two distinct independent triples, each the neighborhood of at least two degree-three vertices.*

Proof. As G is nonplanar and V_8 is toroidal, Halin's theorem gives a three-cut X leaving at least three components. If every H_D is planar, every component of $G - X$ is a single vertex with neighborhood X (Lemmas 3.1 and 3.2), so X is independent by Lemma 3.4(a) and $G = K_{3,k}$. By Ringel's formula [25], $K_{3,k}$ has genus $\lceil (k-2)/4 \rceil$, so $k \geq 7$ and minimality gives $G \cong K_{3,7}$.

Since $G \not\cong K_{3,7}$, some component must have nonplanar triangle completion. Choose a three-cut X leaving at least three components and a component D with H_D nonplanar, minimizing $|V(D)|$. Lemma 3.5, with $Z = \varnothing$,

gives a three-cut A_1, at least two single-vertex components with neighborhood A_1, and one further component D' with $H_{D'}$ nonplanar.

Let W be the set of all vertices with neighborhood A_1. Every vertex of W is a single-vertex component of $G - A_1$. Since $H_{D'}$ is nonplanar, D' has more than one vertex, so $V(D') \cap W = \varnothing$. Now, choose a three-cut X leaving at least three components and a component D disjoint from W with H_D nonplanar, again minimizing $|V(D)|$. Such a choice exists, as $G - A_1$ has at least three components, and D' is one of them, is disjoint from W, and has nonplanar triangle completion. Lemma 3.5, now with $Z = W$, gives a three-cut A_2 and at least two vertices in D with neighborhood A_2. These vertices lie outside W, so their neighborhood cannot be A_1. Thus $A_2 \neq A_1$. Both triples are independent by Lemma 3.4(a). □

An independent triple that is the neighborhood of at least two twins is a *twin triple*. Two distinct twin triples share at most two vertices, so the twin triples of Theorem 3.6 give three cases (Figure 3). If some two twin triples are disjoint, Theorem 4.1 applies. If no two are disjoint but some two share two vertices, Theorem 5.1 applies. Otherwise every two distinct twin triples share exactly one vertex, and Theorem 6.1 applies. In each case we fix two twin triples and two twins with each neighborhood, and write J_1, J_2 for their twin subgraphs. In Section 4, three paths joining J_1 to J_2 give the graph Ψ_1. In Sections 5 and 6, we use their union as Ψ_2 and Ψ_3, respectively. We study the rest of G through bridges. Sections 3.1–3.3 develop the tools used for these bridges.

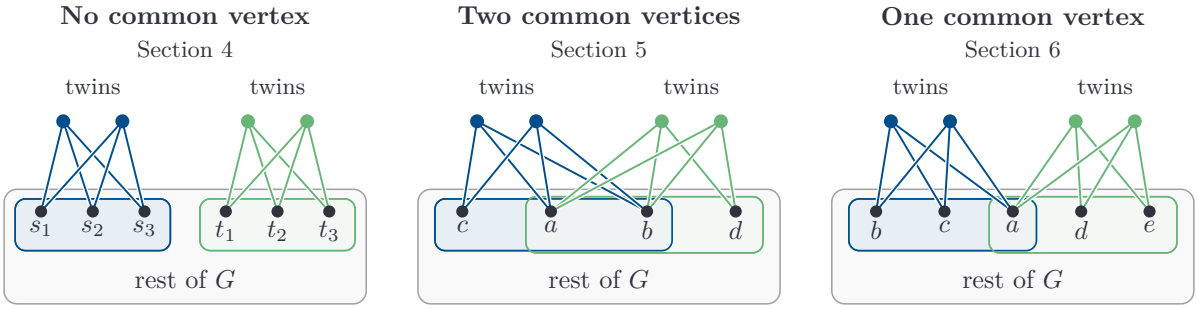


Figure 3: An illustration of the three cases.

3.1 Bridges with three attachments

This subsection studies bridges with three attachments. Let B be a graph with a set A of *attachment* vertices, no two of them adjacent. We assume that its interior is nonempty and connected, and every attachment has a neighbor in it. Every bridge of G with a nonempty interior has these properties.

Adding edges between attachments to B gives a *completion* of B, and when $|A| = 3$, the completion $B \cup K_3[A]$ is its *triangle completion*. An *allowed operation* deletes an interior vertex or an edge, or contracts an edge with an interior end and deletes any edges this creates between attachments. The result must retain the labeled attachments, a nonempty connected interior, and an interior neighbor at every attachment. Graphs obtained by these operations are *allowed minors*. Planar completions remain planar under allowed operations, and replacing a bridge of G by an allowed minor gives a minor of G.

Lemma 3.7. *Let B be a bridge of G with nonempty interior, and let $\mathcal{E}$ be a set of sets of edges between its attachments. Suppose that, for every allowed minor B' of B, whether the graph obtained from G by replacing B with B' is toroidal depends only on which of the completions $B' + E$ with $E \in \mathcal{E}$ are planar. Then every proper allowed minor B' of B has $B' + E$ planar for some $E \in \mathcal{E}$ with $B + E$ nonplanar.*

Proof. Otherwise $B' + E$ is planar exactly when $B + E$ is, for every $E \in \mathcal{E}$. Replacing B with B' then gives a nontoroidal graph, since G is nontoroidal. This is a proper minor of G, contrary to minimality. □

When the hypothesis of Lemma 3.7 holds, we call the completions $B + E$ with $E \in \mathcal{E}$ the *relevant completions* of B.

The following lemma describes the components left by a three-cut of $B \cup K_3[A]$. When B is a bridge of G, the two components containing no attachment are single vertices with the same neighborhood in G.

Lemma 3.8. *Let B be a graph with attachment set $A = \{x, y, z\}$ as above, and suppose $B \cup K_3[A]$ is three-connected and $B + E$ is planar for some $E \subseteq E(K_3[A])$. Let X be a three-cut of $B \cup K_3[A]$ such that $(B \cup K_3[A]) - X$ has at least three components. Then $X \neq A$, and the following hold.*

(i) $(B \cup K_3[A]) - X$ *has exactly three components,* D_0, D_1, D_2*, with* $V(D_0) \supseteq A - X$ *and* $V(D_1), V(D_2) \subseteq \mathrm{Int}(B)$.

(ii) *Every component of* $B - X$ *inside* D_0 *meets* A.

(iii) *For* $i = 1, 2$*,* $N_B(D_i) = X$ *and* $B[V(D_i) \cup X] \cup K_3[X]$ *is planar.*

(iv) *No bridge on* X *in* $B + E$ *other than those of* D_1, D_2 *is adjacent to all of* X.

(v) *If* B *is a bridge of* G *on* A*, then* D_1 *and* D_2 *are single vertices* p *and* q *with neighborhood* X.

(vi) *For every* $E' \subseteq E(K_3[A])$*, a bridge on* X *in* $B + E'$ *containing an edge of* E' *has at least two attachments. Also,* $B + E'$ *is nonplanar exactly when such a bridge is adjacent to all of* X*, or has two attachments* α, β *and becomes nonplanar on adding* $\alpha\beta$*, and in the latter case the completed bridge is a minor of* $B \cup K_3[A]$*. In particular, adding an edge between vertices of* $A \cap X$ *does not change whether or not it is planar.*

If $B \cup K_3[A]$ *is nonplanar and has no* K_5 *minor, such a three-cut* X *exists.*

Proof. Deleting A from $B \cup K_3[A]$ leaves $B[\mathrm{Int}(B)]$, which is connected, so $X \neq A$. Since $|A| = |X| = 3$, $A - X$ is nonempty and is a clique in $B \cup K_3[A]$, so it lies in one component D_0 of $(B \cup K_3[A]) - X$.

Every other component of $(B \cup K_3[A]) - X$ avoids A. The added edges of $B + E$ and $B \cup K_3[A]$ join attachments, so the graphs $B - X$, $(B + E) - X$ and $(B \cup K_3[A]) - X$ have the same components avoiding A. A component of $B - X$ avoiding A is therefore a component of $(B \cup K_3[A]) - X$ other than D_0. Hence every component of $B - X$ inside D_0 meets A, proving (ii).

Let D be a component of $(B \cup K_3[A]) - X$ other than D_0. Three-connectivity of $B \cup K_3[A]$ gives $N_B(D) = X$, so D is a component of $(B + E) - X$ whose bridge on X is adjacent to all of X. By Lemma 2.5(i), the planar graph $B + E$ has at most two such bridges. Since $(B \cup K_3[A]) - X$ has at least three components, there are exactly two, D_1 and D_2, which proves (i) and (iv). By Lemma 2.5(ii), the bridge of each D_i plus $K_3[X]$ is planar, and this graph is $B[V(D_i) \cup X] \cup K_3[X]$, proving (iii). For (v), suppose B is a bridge of G on A. It is planar since $B + E$ is, and D_1, D_2 are components of $B - X$ disjoint from A, so Lemma 3.3 makes them single vertices with neighborhood X. For (vi), suppose a bridge on X in $B + E'$ containing an edge of E' has at most one attachment. Then its interior contains an end of that edge and has at most two neighbors in $B \cup K_3[A]$, consisting of at most one attachment and at most one further vertex of A. This contradicts three-connectivity. Every bridge not containing an edge of E' is unchanged from B, which is planar since $B + E$ is. If such a bridge has two attachments α, β, an α–β path through D_1 makes it plus $\alpha\beta$ a minor of B, so this completion is also planar. As B is planar, Lemma 2.5(i) allows no such bridge other than those of D_1, D_2 to be adjacent to all of X. The bridges of D_1, D_2 are adjacent to all of X by (iii), so Lemma 2.5(i),(ii) makes $B + E'$ nonplanar exactly when a bridge containing an edge of E' is adjacent to all of X, or has two attachments α, β and becomes nonplanar on adding $\alpha\beta$. In the latter case, the completed bridge is a minor of $B \cup K_3[A]$, using an α–β path through D_1.

Finally, if $B \cup K_3[A]$ is nonplanar and has no K_5 minor, Halin's theorem gives such a three-cut, since this graph contains a triangle. □

When adding two edges between the attachments leaves a bridge planar but its triangle completion is nonplanar, we use the next lemma to determine its structure and find certain small allowed minors of the bridge. These are the three graphs in Figure 4, each with attachments x, y, z and all other vertices interior. Renaming the attachment vertices, we will write $T(a)$, $S(a; b)$ and $Q(a; b, c)$ for any distinct attachments a, b, c.

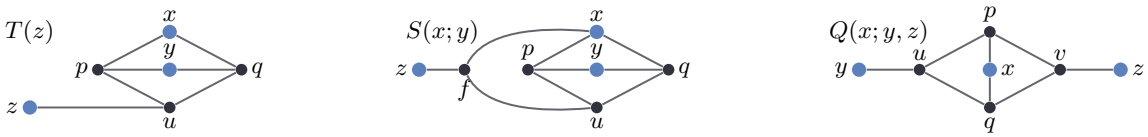


Figure 4: The graphs $T(z)$, $S(x; y)$ and $Q(x; y, z)$.

Lemma 3.9. *Let* B *be a bridge of* G *on* $A = \{x, y, z\}$*, and suppose* $B + xy + xz$ *is planar. Suppose the triangle completion* $B \cup K_3[A]$ *is three-connected, nonplanar and without a* K_5 *minor. By Lemma 3.8, this graph has a three-cut leaving at least three components. Let* X *be any such cut, and let* D_0*,* p *and* q *be as in that lemma with* $E = \{xy, xz\}$*. Then one of the following holds, where* $u, v \in \mathrm{Int}(B)$.

(i) $X = \{x, u, v\}$*. Up to interchanging* u, v*, the graph* $B[V(D_0)]$ *has exactly two components, one containing* y *and adjacent to* u *but not* v*, and one containing* z *and adjacent to* v *but not* u*. Moreover,* $B + xy + yz$ *and* $B + xz + yz$ *are nonplanar, and* B *has* $Q(x; y, z)$ *as an allowed minor.*

(ii) $X = \{x, y, u\}$. *No vertex of* $D_0 - z$ *is adjacent to* y, *and every component of* $B[V(D_0) \setminus \{z\}]$ *has neighborhood* $\{z, x, u\}$ *in* B. *If* $V(D_0) = \{z\}$, *then* $B = T(z)$. *Otherwise,* $B + xy + yz$ *and* $B + xz + yz$ *are nonplanar, and* B *has* $S(x; y)$ *as an allowed minor.*

(iii) $X = \{x, z, u\}$. *The claims of (ii) hold with* y, z *interchanged.*

Proof. Applying Lemma 3.8 with $E = \{xy, xz\}$, the components of $(B \cup K_3[A]) - X$ are $D_0, \{p\}, \{q\}$, with $A - X \subseteq V(D_0)$ and $X \neq A$. The connections from D_0 to the vertices of X necessarily exist in $B \cup K_3[A]$, since $N_{B \cup K_3[A]}(D_0) \subseteq X$ and, by three-connectivity, $N_{B \cup K_3[A]}(D_0) = X$. However, they need not be edges of B.

By Lemma 3.8(v), p, q have neighborhood X, so X is independent by Lemma 3.4(a). Suppose for contradiction that $x \notin X$. We first show that D_0 is also a component of $(B + E) - X$. Since $x \in A - X \subseteq V(D_0)$, it is enough to connect each $w \in V(D_0)$ to x within $(B + E) - X$. By Lemma 3.8(ii), the component of w in $B - X$ contains an attachment in A. If this attachment is x, we are done. Otherwise it is y or z, and the edge xy or xz connects w to x. Also, $N_{B+E}(D_0) = X$, since edges of B are retained and every attachment in X is joined to $x \in V(D_0)$. Thus the bridge with interior $V(D_0)$ is a third bridge adjacent to all of X, contradicting planarity by Lemma 2.5(i). Hence $x \in X$, and since $X \neq A$, its intersection with A is $\{x\}$, $\{x, y\}$ or $\{x, z\}$, with $X - A \subseteq \mathrm{Int}(B)$.

In case (i), $X = \{x, u, v\}$. We have shown that $x \notin X$ gives a $K_{3,3}$ minor of $B + xy + xz$. Interchanging x with y or z shows that $B + xy + yz$ and $B + xz + yz$ are nonplanar, since $y, z \notin X$. Now, if $B[V(D_0)]$ were connected, its bridge on X in $B + xy + xz$ would be adjacent to x through xy and to u, v through edges of B. This would form a third bridge adjacent to all of X, contradicting planarity by Lemma 2.5(i). Since $B[V(D_0)] + yz$ is connected, $B[V(D_0)]$ therefore has exactly two components, D_y containing y and D_z containing z. Each of D_y, D_z is adjacent to x in $B + xy + xz$, so neither can be adjacent to both u, v, as this would give a third bridge adjacent to all of X. If D_y were adjacent to neither, its neighborhood in $B \cup K_3[A]$ would be contained in $\{x, z\}$, contradicting three-connectivity. The same argument applies to D_z with $\{x, y\}$. Thus each is adjacent to exactly one of u, v. Since D_0 is adjacent to both, after interchanging u, v if necessary, D_y is adjacent to u but not v, and D_z to v but not u.

There is a path from u to y through D_y, and a path from v to z through D_z. Contract their internal vertices into u and v, respectively, retaining the edges uy, vz. Delete any remaining vertices of $D_y - \{y\}$ and $D_z - \{z\}$, starting with those furthest from u or v, respectively, with distances measured within the interior, to keep the interior connected. Delete any edges xu or xv that may have been formed. The graph that remains is $Q(x; y, z)$. This minor is allowed, since the interior remains nonempty and connected, the attachments remain distinct, and each keeps an interior neighbor, as x keeps p, q, and y, z keep u, v.

Now consider case (ii), $X = \{x, y, u\}$. Each added triangle edge has an end in X, so

$$(B \cup K_3[A]) - X = B - X,$$

and D_0 is a component of $B - X$. We claim that no vertex of $D_0 - z$ is adjacent to y. If some vertex of $D_0 - z$ were adjacent to y, then D_0 already has a neighbor at u, since its neighborhood in the triangle completion is $X = \{x, y, u\}$ and none of the added triangle edges touches u. The added edge xz makes D_0 adjacent to x. Contracting D_0 to a single vertex m gives a $K_{3,3}$ minor with parts $\{x, y, u\}$ and $\{p, q, m\}$, contradicting the planarity of $B + xy + xz$.

These two facts give the required neighborhoods as follows. If $D_0 - z$ is empty, then $N_{B \cup K_3[A]}(D_0) = X$ forces $uz \in E(B)$, and since X is independent, $B = T(z)$. Otherwise, choose a connected component C of $D_0 - z$. Inside D_0, z is the only possible neighbor of C outside itself. Since D_0 is a component of $B - X$, every neighbor outside D_0 lies in X. No vertex of $D_0 - z$ is adjacent to y, so $N_B(C) \subseteq \{x, u, z\}$. Since C contains no attachments, it has the same neighborhood in $B \cup K_3[A]$. Thus $N_B(C) = \{x, u, z\}$; otherwise, the triangle completion would not be three-connected.

In this nonempty case, the argument for $x \notin X$ also shows that $B + xz + yz$ is nonplanar, since $z \notin X$. In $B + xy + yz$, the connected graph D_0 is adjacent to x, u through edges from C and to y through yz. Contracting D_0 again gives a $K_{3,3}$ minor, so this graph is also nonplanar. Now contract C to a single vertex f and delete the other components of $D_0 - z$, starting with vertices furthest from u in the interior, and keeping the edges fx, fu, fz. Delete uz if it is present. These operations retain all three attachments and keep the interior nonempty and connected. Also, x, y retain their neighbors p, q, and z retains its neighbor f, so the operations are allowed. The resulting graph is $S(x; y)$.

In case (iii), $X = \{x, z, u\}$, and the same argument applies with y, z swapped. □

We next determine which completions of T, S and Q are planar. If a bridge B has one of these graphs as an allowed minor, then every nonplanar completion of that minor gives a nonplanar completion of B.

Lemma 3.10. *Let* E *be a set of edges between* x, y, z. *Then* $T(z) + E$ *is planar if and only if* E *does not contain both* xz *and* yz, $S(x; y) + E$ *is planar if and only if* $yz \notin E$, *and* $Q(x; y, z) + E$ *is planar if and only if* $E \subseteq \{xy, xz\}$ *or* $E = \{yz\}$.

Proof. Adding edges preserves nonplanarity, and deleting edges preserves planarity. Thus it suffices to check the smallest sets E giving nonplanarity and the largest giving planarity. We use the drawing in Figure 4.

For $T(z)$, adding xz, yz gives a $K_{3,3}$ with parts $\{p, q, z\}$ and $\{u, x, y\}$. On the other hand, adding xy inside the four-cycle $pxqy$ and xz in the outer face gives a plane drawing of $T(z) + xy + xz$. Interchanging x, y gives one of $T(z) + xy + yz$. Thus the completion is planar exactly when E does not contain both xz, yz.

For $S(x; y)$, adding yz gives a $K_{3,3}$ subdivision with parts $\{p, q, f\}$ and $\{u, x, y\}$, where the connection from f to y is the path fzy. Adding xy inside $pxqy$ and xz in the outer face gives a plane drawing of $S(x; y) + xy + xz$. Thus the completion is planar exactly when $yz \notin E$.

For $Q(x; y, z)$, adding xy, yz gives a $K_{3,3}$ subdivision with parts $\{p, q, y\}$ and $\{u, v, x\}$, where the connection from y to v is the path yzv. Interchanging y, z and u, v shows that $Q(x; y, z) + xz + yz$ is also nonplanar. Adding yz alone in the outer face gives a plane drawing. To draw $Q(x; y, z) + xy + xz$, redraw the path pxq outside the four-cycle $upvq$, beside u. Put y inside the four-cycle $xpuq$ and z outside both cycles, retaining the edges uy, vz. Then add xy inside $xpuq$ and xz in the outer face. Thus the completion is planar exactly when $E \subseteq \{xy, xz\}$ or $E = \{yz\}$. □

The next proposition describes a bridge with three attachments and a planar completion by two edges, as in Figure 5.

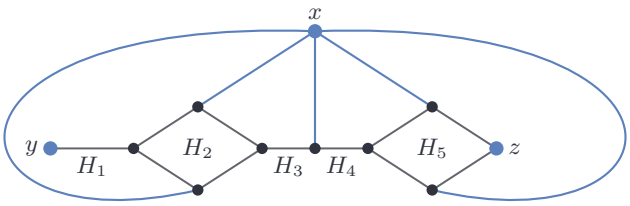


Figure 5: A bridge B as in Proposition 3.11, where $B - x$ has five blocks. The attachments and the edges incident with x are blue.

Proposition 3.11. *Let B be a bridge of G on $A = \{x, y, z\}$ whose triangle completion is three-connected and a minor of G, and suppose $B + xy + xz$ is planar. Then the following hold.*

(i) *The blocks $H_1, \dots, H_n$ of $B - x$ are edges or four-cycles, where y lies only in H_1, z only in H_n, consecutive blocks share one vertex and other blocks are disjoint.*

(ii) *Each four-cycle H_j is a cycle $apbqa$ with $a = y$ or $a \in V(H_{j-1})$, and $b = z$ or $b \in V(H_{j+1})$. The vertices p, q have degree three in B and are adjacent to x.*

(iii) *No three consecutive blocks are edges.*

(iv) *The neighbors of x in B are the vertices p, q of the four-cycles and the vertices shared by two edge blocks.*

(v) *With no four-cycle, B consists of one interior vertex adjacent to x, y, z.*

We say that the blocks form a chain if they can be ordered so that consecutive blocks share exactly one vertex and nonconsecutive blocks are disjoint.

Proof. We prove (i) and the first sentence of (ii) by induction on $|\operatorname{Int}(B)|$. Write $H = B + xy + xz$. We are given that H is planar. The triangle completion $B \cup K_3[A] = B + xy + xz + yz$ has just one more edge and could be planar or nonplanar.

Suppose it is planar. If $G - A$ were connected, it would be the interior of B, so G would be a subgraph of the planar graph $B \cup K_3[A]$, which is impossible.

Thus $G - A$ is disconnected, so A is a three-cut. By Lemma 3.2, the interior of B is a single vertex w, and every attachment is adjacent to w. Hence B is the three-edge star with edges wx, wy, wz. Deleting x leaves the path $y - w - z$, as desired.

Now suppose $B \cup K_3[A]$ is nonplanar. It is three-connected and has no K_5 minor, since it is a minor of G. By Lemmas 3.8 and 3.9, it has a three-cut X leaving at least three components, there are two interior vertices p, q whose neighborhoods are exactly X, and X has one of the three forms $\{x, u, v\}, \{x, y, u\}, \{x, z, u\}$. In the three cases, set $(a, b) = (u, v), (y, u), (u, z)$, respectively, so that $X = \{x, a, b\}$. This set is independent by Lemma 3.4(a). Let D_0 be the remaining component of $(B \cup K_3[A]) - X$, as in Lemma 3.8.

After deleting x, both p, q remain adjacent to a, b, and the paths apb, aqb form a four-cycle $C = apbqa$. Now consider $(B - x) - V(C)$. This removes exactly $X \cup \{p, q\}$, so, by Lemma 3.8, the other vertices belong to D_0, and

$$(B - x) - V(C) = B[V(D_0)].$$

In case (i) of Lemma 3.9, $B[V(D_0)]$ has exactly two components D_y, D_z containing y, z, respectively, with $N_{B-x}(D_y) = \{u\}$ and $N_{B-x}(D_z) = \{v\}$. Thus the four-cycle is a whole block of $B - x$.

In case (ii), $X = \{x, y, u\}$. Deleting X removes all triangle edges, so $(B \cup K_3[A]) - X = B - X$, and D_0 is a component of $B - X$. Hence $B[V(D_0)]$ is connected. Every edge leaving D_0 goes to x, y, u, so after deleting x its only possible neighbors outside itself are y, u. No vertex of $D_0 - z$ is adjacent to y, and $yz \notin E(B)$, since y, z are attachments. Thus there is no edge to y. Since D_0 has neighborhood X in the triangle completion and none of the added triangle edges touches u, its connection to u must be an edge of B. Hence $N_{B-x}(D_0) = \{u\}$. For case (iii), interchange y, z.

Thus the four-cycle is always a block of $B - x$. The remaining graph attaches through two separate parts: one contains y and meets the cycle only at a, and the other contains z and meets the cycle only at b. We now use induction on these parts.

If $y \neq a$, let H_L consist of the component of $(B - x) - V(C)$ containing y, together with a and its edges to that component. Otherwise let H_L be the single vertex y.

If z is outside C, take the component of $(B - x) - V(C)$ containing z, add the vertex b together with all its edges to that component, and call the resulting graph H_R. If $b = z$, let H_R be the single vertex z. The graphs H_L, H_R are disjoint, meet C only at a, b, respectively, and their union with C is $B - x$.

Consider H_R with $b \neq z$. If H_R has only the vertices b, z, then it is connected and hence is the single edge bz.

Otherwise, consider a connected component D of $H_R - \{b, z\}$. Every edge of H_R from D to a vertex outside D ends at b or z. Thus $N_B(D) \subseteq \{x, b, z\}$. Every vertex of D is interior to B, so D has no additional neighbors outside B, and

$$N_G(D) = N_B(D) \subseteq \{x, b, z\}.$$

Three-connectivity of G gives equality, since p lies outside $V(D) \cup \{x, b, z\}$. Let B' be the bridge of G on $\{x, b, z\}$ with interior $V(D)$.

There are at most two such components, since three would give a $K_{3,3}$ minor of H. With two, contracting either into b creates xb, bz, making the triangle completion of the bridge formed from the other a planar minor of H. Lemma 3.2 makes both components single vertices. Their neighborhoods are independent by Lemma 3.4(a), so bz is absent and H_R is a four-cycle with b, z opposite. With one component and the edge bz, the paths xpb, xz, bz similarly give a planar triangle completion of B'. Then D is a single vertex whose neighborhood contains the edge bz, a contradiction. Thus with one component, bz is absent and $H_R = B' - x$.

In this last case, the paths xpb, xz give $B' + xb + xz$ as a planar minor of H. An a–y path in H_L, together with bqa and yz, gives a b–z path in $B \cup K_3[A]$ internally disjoint from those paths and avoiding D. These three paths give the triangle completion of B' as a minor of $B \cup K_3[A]$, hence of G. It is three-connected by Lemma 3.1, and B' has fewer interior vertices than B, since it omits p, q. Induction describes the blocks of $H_R = B' - x$. Interchanging y, z and a, b gives the argument for H_L. Joining their blocks to C proves (i) and the first sentence of (ii).

Every interior vertex has degree at least three in B. The vertices p, q of each four-cycle and those shared by two edge blocks have degree two in $B - x$, so each has degree three and is adjacent to x. Since $N_G(p) = \{x, a, b\}$ is independent, $xa, xb \notin E(B)$. This accounts for every vertex except y, z, and $xy, xz \notin E(B)$, proving (ii) and (iv).

Three consecutive edge blocks would give adjacent degree-three vertices s, t, both adjacent to x. Then s has adjacent neighbors x, t, contradicting Lemma 3.4(a). This proves (iii). With no four-cycle, (iii) allows at most two edge blocks, and $yz \notin E(B)$ rules out one. Thus $B - x$ is the path ywz, and (iv) gives (v). □

3.2 Nonplanar completions

By Lemma 3.7, no proper allowed minor of a bridge of G keeps all its nonplanar relevant completions nonplanar. So if a bridge of G has an allowed minor that does, the bridge must be equal to it.

Lemma 3.12. *Contracting each attachment path of a tripod into its end in the attachment triple A, and each of the six paths of its $K_{2,3}$ subdivision to an edge, gives two vertices with neighborhood A.*

Lemma 3.13. *Let B and $A = \{x, y, z\}$ be as in Section 3.1, with $B \cup K_3[A]$ three-connected and nonplanar. Suppose B lies in a graph with no K_5 minor that also contains three paths joining the pairs of vertices of A, internally disjoint from each other and from B. Then B has $T(r)$ as an allowed minor for some $r \in A$.*

Proof. Since $B \cup K_3[A]$ is nonplanar, it has no embedding in a closed disk with its triangle C on A as the boundary. Applying Theorem 2.8, we see that outcome (a) would require A to have four vertices. Also, the auxiliary graph $\mathrm{Aux}(B \cup K_3[A], C)$ is three-connected, since $B \cup K_3[A]$ is and the added vertex has three neighbors. Thus it has no three-connected component other than the one containing C, so outcome (c) cannot occur. Therefore $(B \cup K_3[A]) - E(C) = B$ contains a tripod.

Call the two degree-three vertices of the tripod's $K_{2,3}$ subdivision p, q. Suppose all three of its attachment paths were trivial, so that x, y, z are the degree-two vertices of this subdivision. Contract each of its six paths to a single edge, placing all contractions in $\mathrm{Int}(B)$. Then p, q are each adjacent to x, y, z. Since $B[\mathrm{Int}(B)]$ is connected, it remains so after these contractions, so it must still contain a p–q path that avoids x, y, z. Shorten this path to an edge, and do the same with the three paths joining the pairs of vertices of A that exist by hypothesis. This gives a K_5 minor, a contradiction.

Thus there is at least one nontrivial attachment path. After relabeling A, let it end at z. Contract the rest of the interior of B onto the tripod and delete the edges outside it, so that every later step keeps the interior connected and adjacent to all of A. Shorten the chosen attachment path to an edge uz, contract the other two attachment paths into their ends x, y, and shorten the six paths of the $K_{2,3}$ subdivision to edges. The remaining graph has p, q each adjacent to x, y, u, together with the edge uz. This is $T(z)$. □

Lemma 3.14. *Let B and $A = \{x, y, z\}$ be as in Section 3.1, and suppose that*

(i) *$B \cup K_3[A]$ is three-connected and has no K_5 minor,*

(ii) *$B + xy$ is planar, but $B + xy + yz$ and $B + xy + xz$ are nonplanar, and*

(iii) *no proper allowed minor of B keeps $B + xy + yz$ and $B + xy + xz$ nonplanar.*

Then $B = Q(z; x, y)$, or $B[\mathrm{Int}(B)]$ is the cycle $a_1b_1a_2b_2a_1$ with remaining edges za_1, za_2 and either yb_1, yb_2, xa_2 or xb_1, xb_2, ya_2. If $B + xz + yz$ is also nonplanar, only these two four-cycle forms occur.

Proof. Contracting xy makes both nonplanar completions in (ii) the graph

$$H_1 = (B \cup K_3[A])/xy.$$

If it were planar, then by the contrapositive of Lemma 2.7, one of $B + xy + yz$ or $B + xy + xz$ would be planar, a contradiction. Thus H_1 is nonplanar. Call the merged vertex r, so xz, yz become rz. Also,

$$H_1 - rz = (B + xy)/xy$$

is a minor of $B+xy$, hence it too is planar. Thus, as H_1 contains no K_5 minor, it must instead have a $K_{3,3}$ subdivision using rz. Call this subdivision H_2. The vertices r, z are consecutive on one branch path. The only vertices of H_2 that could be attachments after the contraction are r, z, so $V(H_2) - \{r, z\} \subseteq \mathrm{Int}(B)$. Write $H_3 = H_2 - \{r, z\}$. Deleting r, z leaves the remaining corner vertices connected, since deleting an edge or at most two vertices of $K_{3,3}$ cannot disconnect it. Every remaining part of a path between corner vertices contains a corner other than r, z. Therefore H_3 is nonempty, connected, and lies in the interior of B.

For every edge rv incident with r in $H_2 - rz$, choose an edge xv or yv of B that becomes rv when xy is contracted. Contract every component of $B[\mathrm{Int}(B)] - V(H_3)$ onto a neighbor in H_3. Keep the edges of $H_2 - r$, the previously chosen edges, and one additional interior edge at whichever of x, y has no chosen edge, if needed and then delete all other edges.

The remaining interior is H_3, which is connected, and every attachment keeps an interior neighbor, so these operations are allowed. Adding either pair xy, yz or xy, xz, and then contracting xy, yields a graph containing H_2, which is nonplanar. Hence, by assumption (iii), this minor must not be proper. Therefore B consists of $H_2 - r$, the chosen edges, and at most one additional edge at x or y.

We next show that r, z are branch vertices of H_2. If r is not, its edges are rz, rs. Suppose the chosen edge for rs is xs. Then x has only the edge xs, and y has only the additional edge yv. If $v = s$, then $\{z, s\}$ separates $\{x, y\}$ from the rest of $B \cup K_3[A]$, contrary to (i). Otherwise contract xs. The remaining interior is $H_3 - s$, which is connected by the preceding argument, since s, r, z are consecutive on one branch path. The vertex y keeps its neighbor v. Also, sz is absent since H_2 has no triangle, so z loses no neighbor and the merged vertex x retains an interior neighbor of s. Thus the contraction is allowed. After adding either pair of edges and contracting xy, the two-edge path srz of H_2 has been shortened to a single edge, leaving a $K_{3,3}$ subdivision and contradicting (iii).

If z is not a branch vertex, its only edge in B is an edge zt of H_2. Contract zt. The interior remains connected, since r, z, t are consecutive on one branch path. As H_2 has no triangle rzt, no chosen edge at x or y ends at t. One of these attachments could therefore lose its last interior neighbor only if its additional edge ends there, say yt. But then $\{x, t\}$ separates $\{y, z\}$ from the rest of $B \cup K_3[A]$, contrary to (i). The merged vertex z also keeps an

interior neighbor, since t has a neighbor other than z, and that neighbor is not r. Thus the contraction is allowed. After adding either pair of edges and contracting xy, the two-edge path rzt of H_2 has been shortened to a single edge, again leaving a $K_{3,3}$ subdivision and contradicting (iii).

The only vertices of H_2 outside $\mathrm{Int}(B)$ are r, z, and we have shown they are both corner vertices. Therefore every subdivision vertex lies in the interior of B. If it had no interior neighbor, its two neighbors in H_2 would be r, z. Together with the edge rz, this yields a triangle, which a $K_{3,3}$ subdivision cannot have. Otherwise, there is an interior edge incident with the subdivision vertex. Contracting an incident interior edge preserves the connected interior and all attachment neighbors, and shortens the path of H_2 containing this vertex by one edge. Both completions therefore remain nonplanar, contrary to (iii). Thus H_2 is $K_{3,3}$.

Because r, z are adjacent, they lie on opposite sides of the bipartition, say $\{r, a_1, a_2\}$ and $\{z, b_1, b_2\}$. Deleting r, z leaves the cycle $a_1b_1a_2b_2a_1$, and z is joined to a_1, a_2. What remains to determine is how x, y attach to the cycle, based on the earlier chosen edges of B.

If the chosen edges for rb_1, rb_2 are xb_1, yb_2, then no additional edge is needed. The bridge B has the four interior vertices a_1, a_2, b_1, b_2 together with the attachment vertices x, y, z. With those chosen edges its edge set is

$$E(B) = \{a_1b_1, b_1a_2, a_2b_2, b_2a_1, za_1, za_2, xb_1, yb_2\}.$$

Renaming (a_1, a_2, b_1, b_2) as (p, q, u, v) gives the four-cycle $upvq$ and the edges zp, zq, ux, vy of $Q(z; x, y)$.

The other possibility is that the chosen edges were instead from the same attachment. After possibly interchanging x, y, they are yb_1, yb_2. Therefore there is also one additional edge xv from earlier which gives x an interior neighbor. Since v is an interior vertex, $v \in \{a_1, a_2, b_1, b_2\}$.

Suppose $v = b_1$, so that the additional edge is xb_1. Delete yb_1, which is allowed because the interior stays connected, x keeps its neighbor b_1, y keeps b_2, and z keeps its two neighbors. After adding either pair xy, xz or xy, yz and contracting xy to r, the contracted graph still contains $K_{3,3}$. Thus both completions are nonplanar, contradicting condition (iii). The same argument works for $v = b_2$ instead.

Thus v is a_1 or a_2 and without loss of generality, we may call it a_2. The interior of B is exactly the four-cycle $a_1b_1a_2b_2a_1$, z is adjacent to a_1, a_2, y to b_1, b_2, and x only to a_2. There are no other vertices or edges in B. Interchanging x, y gives the second form.

Finally, suppose that $B + xz + yz$ is nonplanar. By Lemma 3.10, B cannot be $Q(z; x, y)$. □

We now use minimality of G to identify the bridges themselves for the following completion tests.

Proposition 3.15. *Let B be a bridge of G on $A = \{x, y, z\}$ whose triangle completion is three-connected, nonplanar and a minor of G. Suppose its relevant completions are $B \cup K_3[A]$, $B + xy + xz$, $B + xy + yz$ and possibly $B + xy$, at least one is planar, and every proper allowed minor of B makes at least one nonplanar relevant completion planar. Then B is $T(x)$, $T(y)$, $T(z)$, $S(x; y)$ or $S(y; x)$, or $B + xy$ is relevant and B is $Q(z; x, y)$ or a four-cycle form of Lemma 3.14.*

Proof. First suppose at least one of $B + xy + xz$ and $B + xy + yz$ is planar. After possibly interchanging x, y, assume that $B + xy + xz$ is planar. By Lemma 3.9, either B is $T(y)$ or $T(z)$, or $B + xy + yz$ is nonplanar and B has an allowed minor $Q(x; y, z)$, $S(x; y)$ or $S(x; z)$.

If B is $T(y)$ or $T(z)$, the conclusion follows. Otherwise, contracting vz into z turns $Q(x; y, z)$ into $T(y)$. Contracting fy into y and deleting the resulting attachment edge xy also turns $S(x; z)$ into $T(y)$. Thus B has an allowed minor M equal to $S(x; y)$ or $T(y)$.

The nonplanar relevant completions of B are exactly $B + xy + yz$ and $B \cup K_3[A]$: the other relevant completions are subgraphs of the planar graph $B + xy + xz$. By Lemma 3.10, $M + xy + yz$ is nonplanar, so $M \cup K_3[A]$ is also nonplanar. Hence M preserves every nonplanar relevant completion. By minimality it cannot be a proper allowed minor, so $B = M$. Restoring the possible exchange of x, y gives $T(x)$, $T(y)$, $T(z)$, $S(x; y)$ or $S(y; x)$.

Now suppose both $B + xy + xz$ and $B + xy + yz$ are nonplanar. Since the triangle completion is also nonplanar and at least one relevant completion is planar, $B + xy$ must be relevant and planar. Any proper allowed minor keeping the two nonplanar completions nonplanar would also keep the triangle completion nonplanar. It would therefore preserve every nonplanar relevant completion, contrary to minimality. This gives condition (iii) of Lemma 3.14 and its other hypotheses already hold. Thus B is $Q(z; x, y)$ or one of the two four-cycle forms. □

When two relevant completions are nonplanar, we look for a reduction of the bridge that keeps both nonplanar. The next lemma shows we can find a $K_{3,3}$ subdivision for which replacing xy by xz, yz still gives a nonplanar graph.

Lemma 3.16. *Let B and $A = \{x, y, z\}$ be as in Section 3.1, with B planar. Suppose $B \cup K_3[A]$ has no K_5 minor and is either three-connected or two-connected with every two-cut containing z and avoiding x, y. If $B + xy, B + xz + yz$ are nonplanar, some $K_{3,3}$ subdivision $H \subseteq B + xy$ uses xy and has $H - xy + xz + yz$ nonplanar.*

Proof. Write $K = B \cup K_3[A]$. Since B is planar, every $K_{3,3}$ subdivision in $B+xy$ uses xy. For any such subdivision H, let s, t be the corner ends of the branch path through xy, and let C be the cycle formed by the four branch paths avoiding s, t. By Lemma 2.6, it suffices to find a choice of H with $z \notin V(C)$, which implies the nonplanarity of the replacement.

First suppose the triangle completion is three-connected. It is nonplanar, so Lemma 3.8 with $E = \varnothing$ gives a three-cut $X \neq A$ and exactly three components D_0, D_1, D_2 of $K - X$, with $V(D_0) \supseteq A - X$. Both D_1, D_2 avoid A and have neighbors at every vertex of X; no other bridge of B on X meets all of X.

Let F be the bridge of $B + xy$ on X containing xy. Lemma 3.8(vi), with $E' = \{xy\}$, gives at least two attachments. Since $B + xy$ is nonplanar, either F has all three attachments, or it has two attachments α, β and $F + \alpha\beta$ is nonplanar.

Suppose first that F has two attachments α, β and $F+\alpha\beta$ is nonplanar. For a contradiction, suppose $z \in V(F)$. Since F already contains xy, it then contains all of A. Let $\mathrm{Int}_X(F) = D$. In $B + xy$, the only neighbors of D outside itself are α, β. Adding the remaining triangle edges cannot introduce a neighbor outside F, since $A \subseteq V(F)$. Thus D has only the two neighbors α, β in the triangle completion, contradicting three-connectivity. Therefore $z \notin V(F)$.

By Lemma 3.8(vi), $F + \alpha\beta$ is a minor of K. It therefore has no K_5 minor and, being nonplanar, contains a $K_{3,3}$ subdivision $J \subseteq F + \alpha\beta$. Adding the edge $\alpha\beta$ adds no vertices, so $z \notin V(F)$ implies $z \notin V(J)$.

We must consider whether J uses the added edge $\alpha\beta$ or already lies in $B + xy$, where $\alpha\beta$ may be absent. Since $\alpha, \beta \in X$, both have neighbors in D_1. If J uses $\alpha\beta$, replace that edge by a path from α to its neighbor in D_1, through D_1 to a neighbor of β, and then to β. All edges of this path lie in B, and its internal vertices lie outside F. Since D_1 avoids z, the resulting subdivision still avoids z. Thus there is a $K_{3,3}$ subdivision $H \subseteq B + xy$ avoiding z. Since $C \subseteq H$, it follows that $z \notin V(C)$.

We now return to the case in which F attaches to all of X. Choose $r_0 \in V(F) - X$ and three paths through $F - X$ from r_0 to the vertices of X, meeting only at r_0. Do the same through D_1, D_2, with starting vertices r_1, r_2, respectively. These paths give a $K_{3,3}$ subdivision H with corner sets $\{r_0, r_1, r_2\}$ and X. Every path lies in $B + xy$, so $H \subseteq B + xy$. Since B is planar, H must use xy.

Since $xy \in E(F)$, it lies on one of the three paths from r_0 to X. Write the other end of this branch path as w. The cycle C consists of the four chosen paths through D_1, D_2 to the two vertices of $X - \{w\}$, and contains no interior vertex of F. If $z \notin X$, then C avoids z, since D_1, D_2 also avoid z, and Lemma 2.6 finishes this case.

Now suppose $z \in X$. Since $X \neq A$, there are two cases: either exactly one of x, y lies in X, or neither does. Suppose first that exactly one does. After interchanging x, y, we have $A \cap X = \{x, z\}$ and $X = \{x, z, u\}$. Since y is the only attachment outside X, Lemma 3.8(ii) says that D_0 is a single component of $B - X$ containing y. Adding xy connects its bridge to x, producing F. Since F meets all three vertices of X, D_0 must already have neighbors at z, u in B. It does not also have a neighbor at x, by Lemma 3.8(iv). Thus, in $B + xz + yz$, the bridge containing yz has attachments $\{z, u\}$, while xz is a separate bridge with attachments $\{z, x\}$. The two added edges therefore belong to distinct bridges, each with two attachments, and their attachment pairs share z. We return to this argument at the end of the next case.

For now, consider the second case, in which neither x nor y lies in X. Then $A \cap X = \{z\}$ and $X = \{z, u, v\}$. If x, y belonged to the same component of $B - X$, adding xy would not change its attachments. Since F meets all of X, that bridge would already meet all of X in B, contrary to Lemma 3.8(iv). Thus x, y lie in different components D_x, D_y. Lemma 3.8(ii) shows that these are the only components in D_0.

Since F meets z, at least one of these components has a neighbor at z in B. Interchange x, y if necessary so that D_y has a neighbor at z. If D_x has neighbors at both u, v, choose three internally disjoint paths through D_x from a vertex r_0 to u, v, x. Continue the path to x through xy and then through D_y to z. This gives three paths through F ending at u, v, z, with xy on the path ending at z. Combine these with the three paths through each of D_1, D_2 from the earlier construction. This gives a $K_{3,3}$ subdivision $H \subseteq B + xy$ whose branch path through xy ends at z. Its relevant cycle C uses D_1, D_2, u, v and avoids z, so Lemma 2.6 finishes this case.

Otherwise, D_x does not have neighbors at both u, v. Since $B \cup K_3[A]$ is three-connected, each of D_x, D_y has a neighbor at at least one of u, v. Thus D_x has a neighbor at exactly one of u, v. Since D_y already has a neighbor at z, Lemma 3.8(iv) prevents it from having neighbors at both u, v, so it too has a neighbor at exactly one. They have neighbors at different ones, since their union has both as neighbors. Label them so that D_x has a neighbor at u and D_y has a neighbor at v.

Then, in $B+xz+yz$, the bridges containing the added edges have attachment pairs $\{z, u\}$ and $\{z, v\}$. We finish this case together with the part of the previous case left above, using the same argument.

Applying Lemma 3.8(vi) with $E' = \{xz, yz\}$ shows that at least one of the two bridges becomes nonplanar when its attachments are joined. Exchange x, y if necessary so that the selected bridge uses xz. Name its attachments z, u and the remaining vertex of X v. The other bridge then has attachments z, v. The selected bridge cannot be

only the edge xz, since joining its attachments would leave it planar. Thus $x \notin X$. Let B_1 be the bridge of B on X containing x. The selected bridge is $B_1 + xz$, and joining its attachments gives the nonplanar graph $B_1 + xz + zu$.

In the first case, the single-edge bridge cannot be selected. The selected bridge is the one containing yz before interchanging x, y. After that interchange it is $B_1 + xz$, and the remaining vertex of X is $y = v$ for the common argument below. Since D_1 is connected in B, avoids A, and has neighbors at every vertex of X, there is a z–u path through D_1 outside the interior of B_1. Shortening this path to the edge zu and taking it together with B_1 shows that $B_1 + zu$ is a minor of B, and hence planar. Also, by Lemma 3.8(vi), $B_1 + xz + zu$ is a minor of $B \cup K_3[A]$. It therefore has no K_5 minor, and, being nonplanar, contains a $K_{3,3}$ subdivision $J \subseteq B_1 + xz + zu$. This subdivision must use xz, since otherwise $B_1 + zu$ would be nonplanar.

Replace xz by the path $x \xrightarrow{xy} y \rightsquigarrow v \rightsquigarrow z$, taking the y–v path through the other bridge and avoiding z; this path is trivial if $y = v$. The y–v subpath also avoids the added edge yz, so it uses only edges of B. Take the v–z path through D_2. If J uses zu, replace it by a z–u path through D_1. These replacement paths are all internally disjoint and internally avoid B_1. Call the resulting subdivision H.

Every remaining edge of J belongs to $B_1 \subseteq B$, and every replacement edge belongs to $B+xy$, so $H \subseteq B+xy$. It uses xy in the path replacing xz. It remains to show that $z \notin V(C)$. Since the branch path containing xy contains z, its opposite cycle C avoids this branch path and both of its ends, and therefore avoids z. Lemma 2.6 finishes the three-connected case, supplying the required subdivision H in both subcases.

Now suppose K is not three-connected. Every two-cut contains z and avoids x, y, so neither x nor y is a cut vertex of $K-z$. Let H_1 be the block of $K-z$ containing xy. It is two-connected, since otherwise $K-z$ would consist only of the edge xy, contrary to the nonempty interior of B. Each component D of $(K - z) - V(H_1)$ has exactly one neighbor w_D in H_1, with $w_D \notin \{x, y\}$. Two-connectivity of K also makes it adjacent to z, so $N_K(D) = \{z, w_D\}$ and $V(D) \cup \{w_D\} \subseteq \mathrm{Int}(B)$.

If there is only one such component D and no edge of B from z to H_1, then $B + xy$ is the union of H_1 and the planar graph $B[V(D) \cup \{z, w_D\}]$, meeting only at w_D. Thus H_1 is nonplanar and contains the required subdivision avoiding z. Otherwise, for each D, another component or an edge from z to H_1, followed by a path in the connected graph $H_1 - xy$, gives a z–w_D path in B avoiding D. Consequently $B[V(D) \cup \{z, w_D\}] + zw_D$ is a planar minor of B. Contract each D into w_D to obtain B_0. These contractions lie within the interior, so it remains connected and every attachment keeps an interior neighbor. For every $E \subseteq E(K_3[A])$, $B_0 + E$ is a minor of $B + E$. Conversely, a plane drawing of $B_0 + E$ can be joined along the edges zw_D to the planar graphs just described. Deleting any added edges then gives a plane drawing of $B + E$. Thus $B_0 + xy$ and $B_0 + xz + yz$ remain nonplanar.

The triangle completion $K_0 = B_0 \cup K_3[A]$ is a minor of K. Any cut of K_0 with at most two vertices would also separate K, since each contracted component has only the neighbors z, w_D, already joined by an edge in K_0. It must therefore be a two-cut containing z. But $K_0 - z = H_1$ is two-connected, so no such cut exists. Apply the three-connected case to B_0 and replace each new edge zw_D used in its subdivision by a path through D. This gives a subdivision in $B + xy$ with the same branch vertices and the same position of z on its branch paths. In particular, z still avoids C, proving the result. □

3.3 Reducing a bridge while keeping a $K_{3,3}$ subdivision

Once a completion is known to be nonplanar, the next lemma reduces any bridge with at most five attachments while keeping a $K_{3,3}$ subdivision of that completion, bounding its number of interior vertices.

Lemma 3.17. *Let B be a graph with an independent set A of attachments, $1 \le |A| \le 5$, such that $B-A$ is nonempty and connected and every vertex of A has a neighbor in $B - A$. Let H_0 be a planar graph with $V(B) \cap V(H_0) = A$, such that every vertex of $H_0 - A$ has all its neighbors in A. Let $H \subseteq B \cup H_0$ be a $K_{3,3}$ subdivision. Then B has an allowed minor B' such that $B' \cup H_0$ contains a $K_{3,3}$ subdivision H' with the following properties.*

(i) *The same vertices of A are corner vertices of H and H'. Corresponding paths between corner vertices, with no corner vertices in their interiors, contain the same vertices of A in the same order.*

(ii) *Every vertex of $B' - A$ lies on H'. On each such path in H', the corner vertices and vertices of A divide the path into segments. Each segment is the edge of H_0 joining its ends, if there is one. Otherwise it has exactly one vertex in its interior if both ends lie in A, and is a single edge if not.*

(iii) *The edges of B' outside H' consist of a minimal set of edges in $B'-A$ connecting the components of $H'-V(H_0)$, and one edge to $B' - A$ from each attachment with no such edge in H'. Deleting any edge of the first set would disconnect $B' - A$.*

In particular, $B' - A$ has at most $|A| + 5$ vertices. If $B_0 \cup H_0$ is planar for every graph B_0 obtained from B by one allowed operation, then $B' = B$, so B itself has this form and bound.

Proof. Take the $K_{3,3}$ subdivision $H \subseteq B \cup H_0$, and divide the nine paths between its corner vertices into segments at their interior vertices that belong to A. Replace any segment whose ends are joined by an edge of H_0 with that edge. This edge cannot occur elsewhere in H. If it were on another path, that path would share both ends of the edge with the original path, but two such paths share at most one vertex. This replacement changes the chosen subdivision, while B and H_0 remain unchanged for now.

Every vertex of $H_0 - A$ has all its neighbors in A, so a segment meeting $H_0 - A$ is either a single edge incident with a corner there, or a two-edge path between attachments. These segments already have the required form and are left unchanged. For each remaining segment in B, if at least one end is not an attachment, contract its interior vertices into that end until the segment is a single edge. If both ends are attachments, contract the vertices inside the segment to one vertex, leaving a two-edge path between the attachments. There is at least one interior vertex in this case because A is independent in B. No two corners or attachments are identified, and their order on each path is unchanged. The resulting subdivision H' therefore satisfies (i) and the segment conditions in (ii).

All these contractions were within $B - A$, so the interior of B stays connected and every attachment keeps an interior neighbor. The subdivision H' must meet that interior, since otherwise it would lie entirely in the planar graph H_0. Every component of the interior outside H' therefore has a neighbor in $H' - V(H_0)$. Contract each such component into one of these neighbors. There may still be edges of B outside H'. Keep all edges of H' belonging to B. Of the remaining edges, keep a minimal set within the interior connecting the components of $H' - V(H_0)$. At each attachment with no retained edge to the interior, keep one such edge as well. Delete every other edge of B to obtain B'. The retained edges keep the interior connected and every attachment adjacent to it throughout the deletions, so every operation is allowed. This proves (ii) and (iii).

For the bound, let i be the number of corners of H' in A. At most $6 - i$ vertices of $B' - A$ are corners. Every other vertex of $B' - A$ lies inside a segment between two consecutive attachments on a branch path, and each such segment contributes at most one vertex. A branch path with k attachments in its interior and j ends in A contains $k + j$ attachments in order, so it has $\max\{k + j - 1, 0\}$ such segments.

The sum of the k's over the nine paths is at most $|A| - i$. If $i = 0$, each path with $k > 0$ contributes only $k - 1$ segments, giving at most $|A| - 1$ segments in all. Together with at most six corners this gives $|A| + 5$. If $i > 0$, each path contributes at most k segments unless both ends lie in A, when it contributes $k + 1$. If r of the i corners lie in one part of $K_{3,3}$, exactly $r(i - r)$ paths have both ends in A. Since $1 \le i \le 5$, we have $r(i - r) \le \lfloor i^2/4 \rfloor \le 2i - 1$. Counting the corners and then the segments gives at most $(6 - i) + (|A| - i) + (2i - 1) = |A| + 5$ vertices of $B' - A$.

If $B' \neq B$, the first allowed operation gives a graph B_0 for which $B_0 \cup H_0$ is nonplanar, since $B' \cup H_0$ is a minor of it and contains H'. Thus the stated planarity assumption forces $B' = B$. □

The preceding results give descriptions for bridges with three attachments and a way to reduce bridges while retaining a $K_{3,3}$ subdivision. The construction below lists the forms that these reductions can produce. Figure 6 shows one of them.

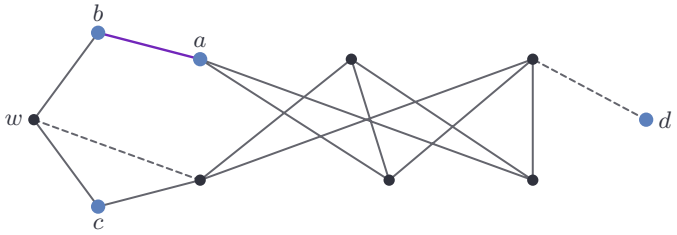


Figure 6: A candidate from Construction 3.18 with attachments a, b, c, d. The edge ab of H_0 is purple, and the edges added in step (iii) are dashed.

Construction 3.18. Let B, A, H_0 be as in Lemma 3.17, and fix A and H_0. The lemma reduces B to an allowed minor B' while leaving H_0 unchanged, so that $B' \cup H_0$ still contains a $K_{3,3}$ subdivision H'. We list candidates that include every possible reduced graph B'.

(i) Begin with the six corner vertices and nine paths of a $K_{3,3}$ subdivision H'. Consider every assignment of the vertices of H_0 to corners, positions inside these paths, or positions outside H', including every possible order along the paths. Each vertex is used only once. Every edge of H' incident with a vertex of $H_0 - A$ must belong to H_0 and join that vertex to an attachment. Every edge of H' between two attachments must also belong to H_0, since A is independent in B'.

Divide the paths into segments at their corner vertices and attachments. If H_0 contains an edge joining the ends of a segment, use that edge. Otherwise, if both ends are attachments, the segment has two edges and

one vertex between its ends. Every other segment is a single edge. Vertices of H' outside H_0 belong to $B' - A$. By Lemma 3.17, there are at most $|A| + 5$ such vertices, so only consider subdivisions satisfying this bound.

(ii) For each subdivision H' satisfying these conditions, make a copy of $H' \cup H_0$. Delete the vertices of $H_0 - A$ and all remaining edges belonging to H_0, retaining every vertex of A, even if it is now isolated. This gives an unfinished candidate for B'. Its interior is nonempty, since otherwise H' would lie entirely in the planar graph H_0.

(iii) Suppose the interior of this candidate has r connected components. Consider every set of $r-1$ edges between these components that makes the interior connected, with every choice of ends. For each resulting graph, join every attachment that still has no interior neighbor to one interior vertex, considering all choices of these neighbors.

Each completed candidate has a nonempty connected interior, every attachment has an interior neighbor, and the attachments remain independent. Also, $B' \cup H_0$ contains the chosen subdivision H'. No vertices were added after (i), so each candidate still has at most $|A| + 5$ interior vertices. Keep one representative of each isomorphism class fixing every attachment in A. The list is finite because A is fixed and the number of interior vertices is bounded. Lemma 3.17 shows that it contains every possible reduced graph B'.

Each application specifies any edges that H' must contain and any further conditions on the subdivision or candidate. Some applications also allow original edges between attachments. In these cases, record separately which edges belong to B and which are supplied by the fixed outside graph or added edges, even when they have the same ends. In (ii), delete only edges supplied from outside B. When the application requires both possibilities, take a segment between attachments either as an original edge or as a path with one interior vertex.

Some applications require several interior components or further components outside the retained subdivision. For these, consider every partition of the interior components obtained in (ii) into the groups specified by the application. Join the components within each group by a minimal set of original edges, considering every allowed choice of ends. If the application allows at most h further components, add up to h single vertices with every allowed choice of attachment neighbors. Then consider every inclusion-minimal set of further original edges that supplies the required paths, connectivity and attachment neighbors. No interior edge may join groups that are required to remain separate. The application specifies which pairs of vertices may be joined, and any additional fixed vertices must also be fixed when comparing candidates for isomorphism. The bound $|A| + 5$ applies to the direct use of Lemma 3.17. In these additional cases, use the reductions and bounds proved in the application.

Some later applications consider a subgraph of G that is not a bridge, or a graph obtained from such a subgraph by adding edges or identifying vertices. Each of these shows that the graph under consideration reduces to one of the listed candidates and that a proper reduction would give a proper nontoroidal minor of G. When vertices were identified for the enumeration, separate them in the ways specified by the application to obtain the candidates for the original subgraph.

In Figure 6, $A = \{a, b, c, d\}$ and H_0 consists of A and the edge ab. In step (i), a is a corner, b, c lie inside the path from a to another corner, and d lies outside H'. The segment from a to b is the edge ab of H_0, the segment from b to c has one vertex w between its ends, and every other segment is a single edge. Step (ii) deletes ab, leaving d isolated and the interior in two components, one of them w alone. Step (iii) joins w to the other component by one edge and joins d to one interior vertex.

The next corollary is the form in which we apply the reduction and the construction to a bridge of G.

Corollary 3.19. *Let B be a bridge of G with nonempty interior and attachment set A, where $1 \le |A| \le 5$, and let H_0 be as in Lemma 3.17, with $B \cup H_0$ nonplanar and without a K_5 minor. Suppose that $B_0 \cup H_0$ is planar for every graph B_0 obtained from B by one allowed operation, or that replacing B in G by an allowed minor B' gives a nontoroidal graph whenever $B' \cup H_0$ is nonplanar. Then B is one of the candidates of Construction 3.18 for A and H_0, and has at most $|A| + 5$ interior vertices.*

Proof. The graph $B \cup H_0$ contains a $K_{3,3}$ subdivision, so Lemma 3.17 gives an allowed minor B' of B for which $B' \cup H_0$ also contains one. Under the first assumption, that lemma gives $B' = B$. Under the second, replacing B by B' gives a nontoroidal minor of G, so $B' = B$ by minimality. The construction lists every such B'. □

Every enumeration in the cases below is recorded with its parameters, candidates, exclusions and counts [Supp. E1–E29]. Proposition 7.1 in Section 7 verifies that $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$ are obstructions and that each graph listed in Theorems 4.1, 5.1 and 6.1 has the properties stated there. We use this throughout without repeatedly referencing it each time.

4 Two twin triples with no common vertex

In this section we consider the case where two twin triples in an obstruction are disjoint. We show that G must then contain a subdivision Ψ_1 of a graph formed from them, and determine G's structure from the Ψ_1-bridges. Denote G's twins as p, q with $N(p) = N(q) = A_1 = \{s_1, s_2, s_3\}$ and twins r, z with $N(r) = N(z) = A_2 = \{t_1, t_2, t_3\}$, where $A_1 \cap A_2 = \varnothing$. Write J_1, J_2 for the twin subgraphs of p, q and r, z, respectively. These ten vertices are distinct, as a vertex adjacent to its twin would be its own neighbor, and twins with different neighborhoods are not adjacent by Lemma 3.4(b). As G is three-connected, Menger's theorem gives, after possibly relabeling A_2, disjoint paths P_i from s_i to t_i meeting $A_1 \cup A_2$ only at their ends. They avoid p, q, r, z, as an internal vertex p, say, would have two neighbors on the path, both in A_1. Choose the paths with minimum total length, and let $\Psi_1 = J_1 \cup J_2 \cup P_1 \cup P_2 \cup P_3$. Then

$$P_i \neq s_i t_i \quad \Longrightarrow \quad s_i t_i \notin E(G), \tag{1}$$

since otherwise replacing P_i by the edge $s_i t_i$ would give a shorter choice. Ψ_1 is a subdivision of two copies of $K_{2,3}$ joined by the matching s_1t_1, s_2t_2, s_3t_3, as illustrated in Figure 7.

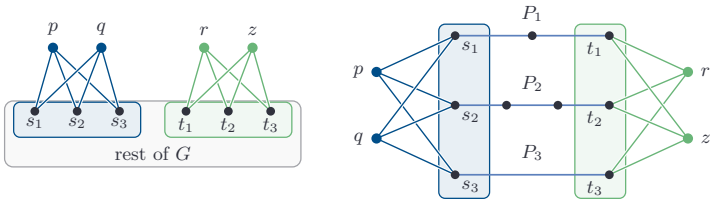


Figure 7: Left, the disjoint twin triples. Right, an example of Ψ_1, with the paths P_i in blue.

Theorem 4.1. *A graph is a three-connected obstruction in which some two twin triples are disjoint if and only if it is isomorphic to one of*

$$\mathfrak{T}_4, \mathfrak{T}_5, \mathfrak{T}_{21}, \mathfrak{T}_{32}, \mathfrak{T}_{38}, \mathfrak{T}_{43}, \mathfrak{T}_{45}, \mathfrak{T}_{48}, \mathfrak{T}_{58}, \mathfrak{T}_{61}, \mathfrak{T}_{63}, \mathfrak{T}_{66}, \mathfrak{T}_{67}, \mathfrak{T}_{68}, \mathfrak{T}_{69}.$$

Section 4.1 finds the faces of Ψ_1 in which the Ψ_1-bridges lie and reduces toroidality to planarity tests. Sections 4.2 and 4.3 restrict the Ψ_1-bridges, and Proposition 4.12 collects the result. Any further vertex with neighborhood A_1 or A_2 gives $G \cong \mathfrak{T}_5$ by Lemma 4.5. Section 4.4 reduces the bridges and bounds their number, and Sections 4.5–4.7 handle the remaining cases.

4.1 Faces and embedding tests of Ψ_1

Any torus embedding of G would restrict to an embedding of Ψ_1, with each Ψ_1-bridge drawn in one of its faces (Section 2.1). We therefore determine the possible face boundaries of Ψ_1 and test whether all its bridges can be drawn together in those faces.

Lemma 4.2. *Every torus embedding of Ψ_1 is cellular with five faces. For some $k \in \{1,2,3\}$, write $\{i,j\} = \{1,2,3\} \setminus \{k\}$. Omitting the twins from the boundary walks, these faces are*

(i) *two faces, each following $s_i P_i t_i, t_j P_j^{-1} s_j$ and meeting no other path.*

(ii) *one face following $s_k P_k t_k, t_y, t_k P_k^{-1} s_k, s_x$, where $x, y \in \{i,j\}$ may be equal.*

(iii) *two other faces meeting the paths exactly at $\{s_k, s_{x'}\}$ and $\{t_k, t_{y'}\}$, respectively, where $\{x, x'\} = \{y, y'\} = \{i,j\}$.*

All twelve choices of k, x and y occur.

Proof. The graph Ψ_1 is connected and nonplanar, as $\Psi_1 - z$ is a subdivision of $K_{3,3}$ with parts A_1 and $\{p, q, r\}$, so its torus embeddings are cellular by Youngs's theorem. Subdividing an edge does not change the faces, so we may take each P_i to be an edge. Up to automorphisms of Ψ_1, there are exactly two torus embeddings [Supp. R1], shown in Figure 8. Their boundary walks, after permuting the indices, give (i)–(iii), with $x = y$ in one and $x \neq y$ in the other. Permuting the indices of both triples simultaneously is an automorphism of Ψ_1, so the first gives the six choices with $x = y$ and the second the six with $x \neq y$. □

We call the faces in (i) the *simple faces*, the face in (ii) the *repeated face*, and those in (iii) the *end faces*, since they meet the paths only at their ends. In each embedding, write $P_{\mathrm{R}} = P_k$ for the *repeated path*, traversed twice by the repeated face boundary walk. Lemma 4.2 shows that internal vertices of P_{R} only appear on the boundary walk

of the repeated face, and internal vertices of the other paths only lie in the boundary walks of the simple faces. A Ψ_1-bridge attaching inside P_R must therefore use the repeated face, and one attaching inside another path must use a simple face. We use the tuple $(P_R; s_x, t_y)$ to record the repeated path and the two additional endpoints on its face. There are three possible repeated paths. For $P_R = P_k$, independently choose $s_x \in \{s_i, s_j\}$ and $t_y \in \{t_i, t_j\}$, giving $3 \cdot 2 \cdot 2 = 12$ combinations of repeated path and additional endpoints.

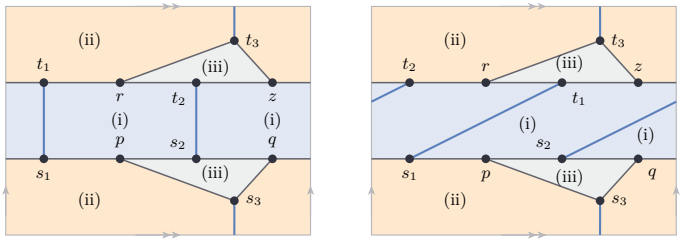


Figure 8: The two torus embeddings of Ψ_1 up to automorphism, with additional endpoints s_1, t_1 (left) and s_1, t_2 (right). Blue marks simple faces (i), tan the repeated face (ii), and gray end faces (iii).

For each choice of repeated path and additional endpoints, we first determine which face boundaries contain each bridge's attachments. We must then check whether the bridges assigned to each face can be drawn there together without crossings. For the simple faces and the repeated face, the next lemma gives planarity conditions.

Lemma 4.3. *Let H be a graph containing Ψ_1 in which p, q, r, z have degree three. For a choice $(P_R; s_x, t_y)$ from Lemma 4.2, write $P_R = P_k$ and $\{i, j\} = \{1, 2, 3\} \setminus \{k\}$. Consider an assignment of each Ψ_1-bridge to the pair of simple faces, the repeated face or one of the end faces, with all its attachments on the corresponding boundary walks.*

Let H_{simp} consist of the bridges assigned to the pair of simple faces together with $(P_i \cup P_j) + s_i s_j + t_i t_j$. Let H_{rep} consist of the bridges assigned to the repeated face together with $P_R + s_k s_x + t_k t_y$. Keep each added edge separate from any bridge edge with the same ends. Then H is toroidal if and only if one of the twelve choices has an assignment satisfying the following conditions.

(a) *H_{simp} is planar.*

(b) *H_{rep} is planar.*

(c) *The bridges assigned to each end face can be drawn together without crossings in a disk, with their attachments on its boundary in the order given by that face's boundary walk.*

If A_1 and A_2 are independent in H and every Ψ_1-bridge with nonempty interior has at least three attachments, no bridge can lie in an end face. Thus H is toroidal exactly when, for one of the twelve choices, the bridges can be assigned to the pair of simple faces and the repeated face with their attachments on the corresponding boundary walks and both H_{simp} and H_{rep} planar.

Proof. Suppose first that H has a torus embedding. Its restriction to Ψ_1 has the faces described in Lemma 4.2 with repeated path $P_R = P_k$ and additional endpoints s_x, t_y. Assign each Ψ_1-bridge to the face containing its drawing away from its attachments (Section 2.1). Since p, q, r, z have degree three in H, no bridge attaches there. Represent each face by a separate closed disk whose boundary traces its boundary walk.

The boundary of each simple facial disk follows both P_i and P_j. Replace its path through p or q by the auxiliary edge $s_i s_j$, and its path through r or z by $t_i t_j$. Both disks now have boundary $C = s_i P_i t_i t_j P_j^{-1} s_j s_i$. Gluing them along their entire boundary C gives a sphere, with C as an equator. The bridge drawings in the two disks give a sphere drawing of H_{simp}, proving (a).

The repeated facial disk has two copies of P_R on its boundary, traversed in opposite directions. Identifying these copies gives an annulus, with s_k, s_x on one boundary component and t_k, t_y on the other (Figure 9). The drawing of P_R joins the two components, and the assigned bridges remain drawn without crossings. Lemma 2.4, applied with these two pairs of boundary vertices, gives the planarity of H_{rep}, proving (b). The bridges assigned to each of the two remaining faces are already drawn together without crossings in that facial disk, with their attachments on its boundary. This gives (c).

Conversely, suppose (a)–(c) hold for one of the twelve combinations $(P_R; s_x, t_y)$. Lemma 4.2 gives a torus embedding of Ψ_1 with this repeated path and these additional endpoints. Let C be the cycle in H_{simp} formed by P_i, P_j and the auxiliary edges $s_i s_j, t_i t_j$. By (a), take a sphere embedding of H_{simp}. Cutting along C gives two disks. The drawing of each bridge assigned to H_{simp} lies in one of these disks, since its drawing away from its attachments is connected. On each disk boundary, replace $s_i s_j$ and $t_i t_j$ by the two-edge paths through the twins on the corresponding simple face boundary. This gives disk drawings for the bridges in the two simple faces. By

Lemma 2.4, the planarity of H_{rep} gives an annulus drawing of P_{R} and the bridges assigned to the repeated face, with s_k, s_x on one boundary component and t_k, t_y on the other. The drawing of P_{R} meets the boundary only at s_k and t_k. Cutting along it gives a disk with two copies of P_{R} on its boundary. Restoring the boundary paths through the twins gives a disk drawing for the repeated face. Together with the disk drawings in (c), these drawings extend the embedding of Ψ_1 to H (Lemma 2.3).

For the last claim, a bridge assigned to an end face has all its attachments in $\{s_k, s_{x'}\}$ or $\{t_k, t_{y'}\}$. It therefore has at most two attachments, both in A_1 or both in A_2. It is not an edge, as A_1 and A_2 are independent, so it has nonempty interior, contradicting that every such bridge has at least three attachments. □

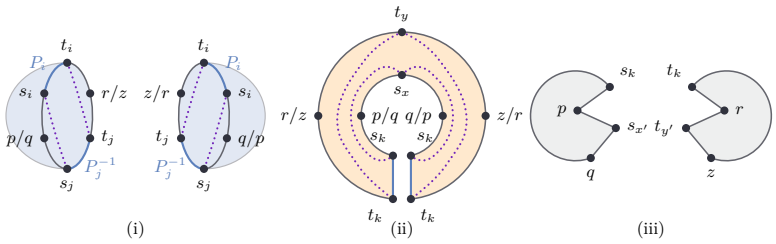


Figure 9: Left, the simple faces. Middle, the repeated face before gluing its two copies of P_{R}. Right, the end faces. Dotted edges replace paths through twins.

4.2 Restrictions on the attachments of Ψ_1-bridges

We now use these tests to restrict the Ψ_1-bridges. For $u, v \in V(P_i)$, write $P_i[u, v]$ for the segment of P_i between them, and $P_i[u, v)$ for this segment without v. For a path P, write $\operatorname{int} P$ for its set of internal vertices.

Lemma 4.4. *Suppose that paths with internal vertices outside Ψ_1 join internal vertices of P_1 to s_2 and to s_3. Then $G \cong \mathfrak{T}_4$. The same is true for t_2, t_3 in place of s_2, s_3 by exchanging A_1 and A_2.*

Proof. If the two paths lie in different bridges, shorten both to edges and contract the segment of P_1 between their ends. If they lie in one bridge, contract its interior into one of its attachments in the interior of P_1. Deleting everything else outside Ψ_1 and shortening the remaining segments gives $P_1 = s_1 u t_1$, the other paths edges, and edges us_2, us_3. This minor is isomorphic to $\mathfrak{T}_4$ [Supp. W2]. Thus, $G \cong \mathfrak{T}_4$. □

Lemma 4.5. *If the attachments of a Ψ_1-bridge with nonempty interior lie on no common face in any torus embedding of Ψ_1, then $G \cong \mathfrak{T}_5$. If $G \ncong \mathfrak{T}_5$, then p, q are the only vertices with neighborhood A_1, and r, z are the only vertices with neighborhood A_2.*

Proof. Let B be the bridge. By Lemma 4.2, every set $V(P_\ell) \cup V(P_m)$ or $V(P_\ell) \cup \{s_m, t_n\}$ with $m, n \neq \ell$ lies on a face boundary in some torus embedding of Ψ_1. So the attachments of B lie in no such set. Let u_i denote an internal vertex of P_i. After permuting the path indices and, if necessary, interchanging A_1, A_2, the attachments contain one of

$$\{u_1, u_2, u_3\}, \qquad \{u_1, u_2, s_3\}, \qquad \{u_1, s_2, s_3\}, \qquad A_1.$$

Indeed, if they include internal vertices of all three paths, the first case holds. If of exactly two, say P_1, P_2, they also include an end of P_3, giving the second case. If of exactly one, say P_1, they must include both s_2, s_3 or both t_2, t_3, giving the third case. If there are no internal attachments, they contain A_1 or A_2. Otherwise, each of A_1, A_2 contains at most two attachments, so all attachments lie in $\{s_\ell, s_m, t_\ell, t_n\}$ for some $\ell, m, n \in \{1, 2, 3\}$ with $m, n \neq \ell$.

Contract the connected interior of B to a vertex w, retain only its edges to the chosen triple, and delete the other Ψ_1-bridges. We show the first three attachment sets cannot occur. In each, contract wu_1. For each u_j in the triple with $j = 2, 3$, contract $P_j[s_j, u_j]$ into s_j. Shortening the remaining segments gives $P_1 = s_1 u_1 t_1$, the other paths edges, and edges $u_1 s_2, u_1 s_3$. This is the $\mathfrak{T}_4$ minor in the proof of Lemma 4.4. It is proper because wu_1 was contracted, and it is nontoroidal, contrary to minimality. Hence the attachments contain A_1. Shortening each P_i to an edge gives Ψ_1 with a vertex w adjacent to A_1, which is $\mathfrak{T}_5$ [Supp. W3]. This minor is nontoroidal, so $G \cong \mathfrak{T}_5$.

Any vertex $w \notin \{p, q\}$ with neighborhood A_1 lies outside Ψ_1, since every vertex of Ψ_1 other than p, q has a neighbor outside A_1. The vertex w and its three incident edges therefore form a bridge attached at A_1, which lies on no common face boundary by Lemma 4.2, so the first part gives $G \cong \mathfrak{T}_5$. The same argument applies to A_2. □

We next use the absence of a K_5 minor to exclude two further attachment possibilities for a Ψ_1-bridge.

Lemma 4.6. *A connected set outside Ψ_1 cannot attach to s_i, s_j, t_i, t_k for three distinct indices i, j, k.*

Proof. Suppose a connected set D outside Ψ_1 is adjacent to s_i, s_j, t_i, t_k, where i, j, k are distinct. Relabel so that $(i, j, k) = (1, 2, 3)$. Retain Ψ_1, the graph induced by D, and one edge from D to each of s_1, s_2, t_1, t_3. Contract D to a vertex x and shorten each P_i to the edge $s_i t_i$. Thus x is adjacent to s_1, s_2, t_1, t_3. The five sets

$$\{p, q, s_1\}, \qquad \{r, s_2, t_2\}, \qquad \{s_3, t_3\}, \qquad \{t_1, z\}, \qquad \{x\}$$

are disjoint and connected, with an edge between every two sets. Contracting each to a vertex gives a K_5 minor. □

Lemma 4.7. *Let $i \neq j$. At most one of $s_i t_j$ and $s_j t_i$ is an edge of G, and neither is an edge if a path with internal vertices outside Ψ_1 joins internal vertices of P_i and P_j. More generally, order P_i and P_j from A_1 to A_2, and let u precede v on P_i and w precede h on P_j. Then there are no disjoint paths from u to h and from v to w with internal vertices outside Ψ_1.*

Proof. Suppose there are disjoint paths P from u to h and P' from v to w, with internal vertices outside Ψ_1. Delete everything outside $\Psi_1 \cup P \cup P'$. Contract $P_i[s_i, u]$ into s_i and $P_i[v, t_i]$ into t_i, and contract $P_j[s_j, w]$ into s_j and $P_j[h, t_j]$ into t_j. Now P joins s_i to t_j, and P' joins t_i to s_j. Shortening P, P' and the three paths of Ψ_1 to edges gives Ψ_1 with every path an edge, together with $s_i t_j$ and $s_j t_i$. Let k be the third index. The five sets

$$\{p, s_k, t_k, z\}, \qquad \{q, s_j\}, \qquad \{r, t_i\}, \qquad \{s_i\}, \qquad \{t_j\}$$

are disjoint and connected, with an edge between every two. Contracting each to a vertex gives a K_5 minor (Figure 10). The edges $s_i t_j$ and $s_j t_i$ would be such paths, with $u = s_i$, $v = t_i$, $w = s_j$ and $h = t_j$, so at most one of them is an edge of G. Now let P_0 join an internal vertex x of P_i to an internal vertex y of P_j, with internal vertices outside Ψ_1. Then P_0 contains none of s_i, t_i, s_j, t_j. So the edge $s_i t_j$ and P_0 would be such paths, with $u = s_i$, $v = x$, $w = y$ and $h = t_j$. Interchanging i and j excludes $s_j t_i$ in the same way. □

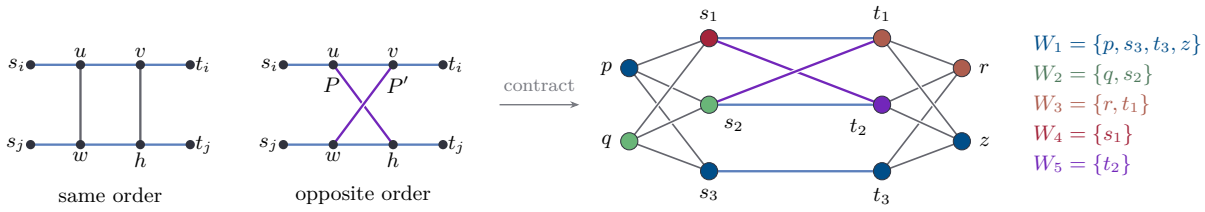


Figure 10: Disjoint paths joining P_i to P_j in the same order and in opposite order.

We next exclude three configurations in which $P_2 = s_2 t_2$, $P_3 = s_3 t_3$, and all Ψ_1-bridge attachments lie in $V(P_1) \cup \{s_2, t_2\}$. In the first two, P_1 is also an edge and there is one bridge, attached at $\{s_1, t_1, s_2, t_2\}$ or $\{s_1, t_1, s_2\}$, respectively. In the third, every bridge attaches in the interior of P_1. The next lemma shows that, in each configuration, a torus embedding of $H/s_3 t_3$ gives one of H.

Lemma 4.8. *Let H contain Ψ_1 and have no K_5 minor, with p, q, r, z of degree three, $P_2 = s_2 t_2$ and $P_3 = s_3 t_3$. Suppose every Ψ_1-bridge has all its attachments in $V(P_1) \cup \{s_2, t_2\}$, and one of the following holds.*

(a) *$P_1 = s_1 t_1$, and there is exactly one Ψ_1-bridge, with attachment set $\{s_1, t_1, s_2, t_2\}$.*

(b) *$P_1 = s_1 t_1$, and there is exactly one Ψ_1-bridge, with attachment set $\{s_1, s_2, t_1\}$.*

(c) *Every Ψ_1-bridge has an attachment in the interior of P_1.*

If $H/s_3 t_3$ is toroidal, then so is H.

Proof. Suppose $H/s_3 t_3$ is toroidal. Take a torus embedding of it and consider the induced embedding on $\Psi_1/s_3 t_3$, which is cellular by Youngs's theorem since this graph is connected and nonplanar. Write $a = s_1$, $b = t_1$, $c = s_2$, and $d = t_2$.

Suppose (a) holds, and let B be the unique Ψ_1-bridge, whose interior must lie entirely in one facial disk F. In [Supp. R2] we record the cyclic sequences of attachment appearances, including repeated appearances, for all possible faces F containing all four attachments in all possible toroidal embeddings of $\Psi_1/s_3 t_3$. Up to cyclic shift and reversal, all such sequences also occur around a face in some torus embedding of Ψ_1, with the exception of a, b, c, d. For those other sequences, the Jordan–Schoenflies theorem allows us to transfer the drawing of B and obtain a toroidal embedding of H.

Suppose otherwise that B is drawn in a face F of the induced embedding with cyclic boundary order a, b, c, d. There exists a simple face of a toroidal embedding of Ψ_1 which has cyclic boundary order a, b, d, c, so we aim to obtain a drawing of B with this order. Put $D = B - \{a, b, c, d\}$. As D is nonempty and connected, with every edge of B having an end in D and every attachment having a neighbor there, Lemma 2.9 implies that, if such a drawing did not exist, there must be vertex-disjoint paths a–d and b–c with nonempty interiors in D. Take a shortest path in D joining their interiors.

Contract the interiors of the first two paths into a, b, respectively, and shorten the joining path to ab. Retaining these edges, $J_1 \cup J_2$ and P_2, P_3 gives Ψ_1 with all three paths edges, together with $ad = s_1t_2$ and $bc = s_2t_1$. The contractions in the proof of Lemma 4.7 give a K_5 minor, a contradiction. Thus the required drawing exists and gives a torus embedding of H.

For case (b), we may use the same argument, restricting to the case where attachment d is unused. Read the facial boundary walks in [Supp. R2] for Ψ_1/s_3t_3 and Ψ_1, recording only appearances of a, b, c. Every such sequence containing a, b, c for the contracted graph also occurs for Ψ_1. In particular, the order a, b, c occurs in the simple face with order a, b, d, c by leaving d unused. Transferring the drawing of the bridge to the corresponding face therefore gives a torus embedding of H. This completes both (a) and (b).

For (c), if there are no bridges, then $H = \Psi_1$ is toroidal. Otherwise every bridge has an attachment to an internal vertex of P_1. Let H_1 consist of P_1 and all such bridges. We combine the faces incident with P_1 in the induced embedding into one region containing the drawing of H_1.

To construct this region, represent each such face by a disk whose boundary follows its facial walk, with repeated appearances of a vertex kept separate. If the two sides of P_1 lie in different faces, identify the copies of P_1 on the boundaries of their two disks. This gives a disk. If both sides lie in the same face, identify the two copies of P_1 on that disk's boundary. This gives an annulus. In either case, keep all other boundary appearances separate. The resulting region contains the drawing of H_1, with P_1 inside it apart from its ends.

As in the previous cases, we consider the possible cyclic sequences of attachment appearances for H_1 in this combined region. We compare them with those which occur in a toroidal embedding of Ψ_1, again with P_1 removed apart from its ends. Putting H_1 into one of these regions would restore P_1 and all the bridges.

If the region is an annulus, the boundary attachment sequences are $a, c \mid b$, $a, c \mid b, d$, $a \mid b$ or $a \mid b, d$. Each also occurs for Ψ_1 by [Supp. R2], so Lemma 2.4 lets us insert H_1 and obtain a torus embedding of H.

If it is a disk, the sequences are a, b, a, b, c, a, b, d, a, b, c, d or a, c, d, b, d, c. For Ψ_1, take the disk drawings of its two simple faces and identify only their boundary copies of P_1. The resulting disk has the last sequence. It also gives a, b, c and a, b, d by using the appearances after b, so only a, b, c, d with all four attachments remains to be considered.

Again we get a K_5 minor if the required drawing in order a, b, d, c does not exist. Put $D = H_1 - \{a, b, c, d\}$. The internal vertices of P_1 connect all the bridge interiors, so D is connected. Every edge of H_1 has an end in D, and every attachment has a neighbor there. As in (a), Lemma 2.9 gives a drawing in order a, b, d, c, or disjoint a–d and b–c paths through D. Joining these through D gives the same K_5 minor as in (a), with the joining path supplying ab. Hence the drawing in order a, b, d, c exists. Using the appearances of d, c after b in the disk above gives a torus embedding of H. □

4.3 Ψ_1-bridges meeting the interior of two paths

We now show that no Ψ_1-bridge can meet the interiors of two of the paths. We first give a planarity test when all bridges attach to $P_1 \cup P_2$ and some bridge meets both interiors.

Lemma 4.9. *Let H consist of Ψ_1 together with some of its bridges in G. Suppose every such bridge has all its attachments in $V(P_1) \cup V(P_2)$, and at least one meets the interiors of both paths. Let H_{simp} consist of P_1, P_2, these bridges, and the edges s_1s_2, t_1t_2. Then H_{simp} has no K_5 minor, and H is toroidal if and only if H_{simp} is planar.*

Proof. Contracting ps_1, rt_1 and deleting q, z and all vertices of P_3 gives H_{simp} from H. Thus H_{simp} is a minor of G and has no K_5 minor. Suppose first that H is toroidal. By Lemma 4.2, a bridge meeting the interiors of P_1, P_2 lies in a simple face for these paths, so P_3 is the repeated path. The repeated face meets $P_1 \cup P_2$ only in s_x, t_y, and each end face meets it in one vertex. Since G is three-connected, every retained bridge with nonempty interior has at least three attachments. Thus only an edge s_xt_y could lie outside the two simple faces. Both P_1, P_2 have internal vertices, so (1) excludes s_1t_1, s_2t_2. The bridge meeting both interiors contains a path between them with internal vertices outside Ψ_1, so Lemma 4.7 excludes s_1t_2, s_2t_1. Hence every bridge lies in a simple face, and Lemma 4.3 gives that H_{simp} is planar. Conversely, suppose H_{simp} is planar. Apply Lemma 4.3 with $(P_{\text{R}}; s_x, t_y) = (P_3; s_1, t_1)$, assigning every bridge to the pair of simple faces. The graph $H_{\text{rep}} = P_3 + s_3s_1 + t_3t_1$ is a path, and no bridges are assigned to the end faces. Thus H is toroidal. □

If every Ψ_1-bridge attaches to $P_1 \cup P_2$ and one meets both interiors, Lemma 4.9 makes H_{simp} nonplanar with no K_5 minor, since G is nontoroidal. By Kuratowski's theorem, H_{simp} therefore contains a $K_{3,3}$ subdivision. The next lemma uses this subdivision and Lemma 3.17 to reduce the possibilities to a finite list and exclude this case.

Lemma 4.10. *If every Ψ_1-bridge has all its attachments in $V(P_1) \cup V(P_2)$, then no Ψ_1-bridge meets the interiors of both paths.*

Proof. Suppose a bridge contains a path P from the interior of P_1 to that of P_2, with internal vertices outside Ψ_1. No bridge attaches to P_3, so minimum degree gives $P_3 = s_3t_3$.

Let H_1 consist of P_1, P_2, and every Ψ_1-bridge, and let $H_2 = (J_1 \cup J_2) + s_3t_3$. Write $A = \{s_1, t_1, s_2, t_2\}$ and $E = \{s_1s_2, t_1t_2\}$. Then $G = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) = A$.

If a component of $G - V(\Psi_1)$ attaches only at path ends, it has at least three attachments in A by three-connectivity, and therefore has either an s_1–t_2 or an s_2–t_1 path disjoint from P. This path along with P contradicts Lemma 4.7. Therefore every such component attaches inside P_1 or P_2, and P makes $H_1 - A$ nonempty and connected. Every vertex of A has an interior neighbor along P_1 or P_2, and A is independent by the minimum-total-length choice of the paths in (1), Lemma 4.7, and Lemma 3.4(a). Thus H_1 is an H_2-bridge with attachment set A.

By Lemma 4.9, $H_1 + E$ is nonplanar with no K_5 minor. Let H_1' be an allowed minor of H_1 for which $H_1' + E$ is nonplanar, and suppose $H_2 \cup H_1'$ has a torus embedding. The connected graph $H_1' - A$ lies in one face of H_2, with all four attachments on its boundary. Since H_2 consists of two copies of $K_{2,3}$ joined by an edge, it is planar. We must therefore allow for an annular face with attachments on both boundary walks, as in Section 2.1.

We consider all possible embeddings of H_2 and a closed regular neighborhood $\mathcal{N}$ of it. Enumerating the rotations of H_2 and tracing their boundary walks gives the following [Supp. R12]. If $\mathcal{N}$ has genus one, no boundary walk contains all four attachments, and every complementary region is then a disk by Lemma 2.1, a contradiction.

If $\mathcal{N}$ has genus zero, no boundary walk contains all four attachments, and exactly one pair contains all four between them. Their attachment sequences are s_1, s_2 and t_1, t_2, with each attachment appearing once. Lemma 2.1 therefore puts $H_1' - A$ in the annulus bounded by these walks, with all four attachments on its boundary. By Lemma 2.4, $H_1' + E$ is planar, a contradiction.

Therefore replacing H_1 by H_1' gives a nontoroidal graph. Corollary 3.19, with attachments A and added edges E, gives the 26,633 possible forms of H_1. Adjoining H_2 to each gives all possible G, and [Supp. E1] excludes them all. Hence no bridge contains such a path. □

We now show further that even allowing bridges to attach outside of only $P_1 \cup P_2$, no Ψ_1-bridge meets the interior of two of the paths P_ℓ.

Lemma 4.11. *No Ψ_1-bridge meets the interiors of two of the paths P_1, P_2, P_3.*

Proof. Suppose a bridge contains a path P from $u \in \operatorname{int} P_1$ to $v \in \operatorname{int} P_2$, with internal vertices outside Ψ_1. By Lemma 4.2, drawing P requires $P_{\text{R}} = P_3$.

No bridge meets both the interior of P_1 and P_3. Otherwise, choose a path P' from $u' \in \operatorname{int} P_1$ to $w \in V(P_3)$, with internal vertices outside Ψ_1. If it lies in a different bridge from P, contract $P_1[u, u']$ into u. Otherwise contract into u the vertices outside Ψ_1 of a tree joining u, v, w. Shortening the resulting paths gives uv, uw. Interchanging A_1, A_2 if necessary, take $w \neq t_3$. Contract $P_2[s_2, v]$ into s_2 and $P_3[s_3, w]$ into s_3. The edges us_2, us_3 give $\mathfrak{T}_4$ by the contractions in the proof of Lemma 4.4. This nontoroidal minor is proper since v is internal to P_2, a contradiction. Interchanging P_1, P_2 also excludes a bridge meeting the interior of P_2 and P_3.

Some bridge meets P_3 by Lemma 4.10. Deleting all such bridges, retaining their attachments, gives a toroidal proper minor. Lemma 4.9 therefore makes the graph H_{simp} formed by P_1, P_2, the remaining bridges and s_1s_2, t_1t_2 planar. By Lemma 4.3, the remaining bridges can therefore be drawn in the simple faces for each of the four pairs of additional endpoints when $P_{\text{R}} = P_3$. The bridges meeting P_3 must use the repeated face, since the simple faces miss P_3 and three-connectivity and independence of A_1, A_2 exclude the end faces by the same lemma. Since G is nontoroidal, these bridges cannot be drawn together there for any of the four pairs. Retain only Ψ_1, P and the bridges meeting P_3. The path P still requires $P_{\text{R}} = P_3$, and these bridges and the repeated-face boundaries are unchanged. The retained graph is therefore nontoroidal, so minimality makes it equal to G. Minimum degree now gives $P = uv$, $P_1 = s_1ut_1$ and $P_2 = s_2vt_2$.

Take a torus embedding of the proper minor G/us_1. The bridges meeting P_3 avoid u, v, so their attachments are unchanged. The only new edges are s_1t_1, s_1v, so A_1, A_2 remain independent, and every bridge with nonempty interior retains at least three attachments. Lemma 4.3 therefore excludes the end faces. With $P_{\text{R}} = P_3$, the repeated face still has the same boundary and bridges, so they still cannot be drawn together there. If $P_{\text{R}} = P_1$, the only faces containing v are the simple faces, which do not contain s_1, so the edge s_1v cannot be drawn. Thus P_2 is repeated, and its repeated face contains s_1.

A bridge meeting P_3 and either s_2 or t_2 must use this repeated face. That face meets P_3 at most at t_3, so the bridge meets P_3 only at t_3. Applying the same argument to G/ut_1 forces it to meet P_3 only at s_3, a contradiction. Interchanging indices $1, 2$ likewise excludes attachments at s_1, t_1. Thus every bridge meeting P_3 attaches only to that path. There is at least one such bridge, so $\{s_3, t_3\}$ separates p from the internal vertices of P_3 and these bridges, contrary to three-connectivity. □

The preceding proofs use minimum total length only through (1). Thus Lemma 4.11 remains valid for any three disjoint paths P_i from s_i to t_i meeting $A_1 \cup A_2$ only at their ends, provided $s_it_i \notin E(G)$ whenever $P_i \neq s_it_i$. We may therefore change the paths later without minimizing their total length, as long as this condition is kept.

We now want to describe the entire connected parts outside $J_1 \cup J_2$, including the internal vertices of the paths, so that we can apply the bridge results of Section 3. Until now, the nonempty interiors of the Ψ_1-bridges were the components of $G - V(\Psi_1)$. We now consider the $(J_1 \cup J_2)$-bridges, whose nonempty interiors are the components of $G - V(J_1 \cup J_2)$. Thus an internal vertex of a path, which could be an attachment of a Ψ_1-bridge, now belongs to the interior of one of these larger bridges. Since the twins have degree three, these are exactly the bridges on $A_1 \cup A_2$ in $G - \{p, q, r, z\}$.

By Lemma 4.11, each nontrivial P_m, together with all Ψ_1-bridges meeting its interior, forms one $(J_1 \cup J_2)$-bridge. It contains no internal vertex of another path, and its attachments are among the six path ends. The other bridges are the Ψ_1-bridges attached only at path ends and those P_i that are single edges. These single-edge paths are $(J_1 \cup J_2)$-bridges, although they were not Ψ_1-bridges. For the bridge containing a nontrivial P_m, let D_m be the graph induced by its interior and put $W_m = N(D_m) - \{s_m, t_m\}$, its attachments other than the ends of P_m. When P_m is an edge, put $W_m = \varnothing$. The next proposition bounds the attachment sets of bridges with nonempty interior and lists the four-attachment cases.

Proposition 4.12. *Suppose $G \not\cong \mathfrak{T}_4$ and $G \not\cong \mathfrak{T}_5$. Let B be a $(J_1 \cup J_2)$-bridge with nonempty interior. Then B has either three or four attachments, and its interior meets at most one of the paths P_i. If B has exactly four attachments, one of the following holds.*

(a) *It lies outside the paths, with attachment set $\{s_x, t_x, s_y, t_y\}$ for some $x \neq y$.*

(b) *It contains P_m, with attachment set $\{s_m, t_m, s_x, t_x\}$ for some $x \neq m$.*

(c) *It contains P_m, with attachment set $\{s_m, t_m, s_x, t_y\}$, where $\{m, x, y\} = \{1, 2, 3\}$.*

Proof. The attachments of B separate its nonempty interior from p, so there are at least three by three-connectivity. If its interior met two of the paths P_i, take a shortest path within it between interiors of distinct P_i. Its internal vertices lie outside Ψ_1, so it lies in a Ψ_1-bridge meeting both path interiors, contrary to Lemma 4.11.

If the interior of B avoids all the paths, then B is a Ψ_1-bridge. Since $G \not\cong \mathfrak{T}_5$, Lemma 4.5 places its attachments on one face boundary in some torus embedding of Ψ_1. By Lemma 4.2, they therefore lie in some $\{s_\ell, t_\ell, s_n, t_k\}$ with $n, k \neq \ell$. Hence there are at most four. With four, Lemma 4.6 forces $n = k$, giving type (a).

Otherwise, B contains P_m. For each $d \in W_m$, connectivity of D_m gives a shortest path from d to the interior of P_m with internal vertices in D_m. These vertices lie outside Ψ_1, since the path first meets P_m at its end and D_m meets no other path. As $G \not\cong \mathfrak{T}_4$, Lemma 4.4 allows at most one vertex of A_1 and one of A_2 in W_m. Together with s_m, t_m, this gives at most four attachments. With four, $W_m = \{s_x, t_y\}$ for $x, y \neq m$. If $x = y$, we have type (b). Otherwise, $\{m, x, y\} = \{1, 2, 3\}$, giving type (c). □

4.4 Bridge reductions and bounds

Apart from $\mathfrak{T}_4, \mathfrak{T}_5$, Proposition 4.12 leaves three cases. If every P_i is an edge, Section 4.5 applies. Otherwise, if no bridge has four attachments and at least two interior vertices, every bridge with at least two interior vertices has three attachments, and Section 4.6 applies. If some bridge has four attachments and at least two interior vertices, it has type (a), (b) or (c), and Section 4.7 applies. In type (b), other twin pairs and paths lead to one of these same cases. From here on, unless stated otherwise, bridges are $(J_1 \cup J_2)$-bridges other than the single-edge paths P_i, and $G \not\cong \mathfrak{T}_4$ and $G \not\cong \mathfrak{T}_5$. By Lemma 4.5, only p, q have neighborhood A_1, and only r, z have neighborhood A_2.

Proposition 4.12 restricts the attachment sets. To use the bridge results of Section 3 to restrict the possible forms of the bridges, we first need their triangle completions to be three-connected and have no K_5 minor.

Lemma 4.13. *Let B be a Ψ_1-bridge with attachment set $A = \{w, s_j, t_k\}$, where $w \in V(P_i)$, and either i, j, k are distinct, or $j \neq i$, $w = s_i$ and $k = i$. Then $B \cup K_3[A]$ is a three-connected minor of G, and if it is nonplanar, B has an allowed minor $T(v)$ for some $v \in A$. This also holds with A_1, A_2 interchanged.*

Proof. First suppose i, j, k are distinct. Join w to s_j along $wP_is_ips_j$, w to t_k along $wP_it_irt_k$, and s_j to t_k along $s_jP_jt_jzt_k$. In the second case, $A = \{s_i, s_j, t_i\}$. The paths s_ips_j, P_i and $s_jP_jt_jrt_i$ join s_i to s_j, s_i to t_i, and s_j to t_i, respectively. In either case, the three paths have disjoint interiors in $\Psi_1 - A$, so they meet B only at their ends. Keeping B and contracting these paths to edges, without identifying their ends, gives $B \cup K_3[A]$ as a minor of G. Since $B[\mathrm{Int}(B)]$ is a component of $G - A$, Lemma 3.1 also gives that $B \cup K_3[A]$ is three-connected. If $B \cup K_3[A]$ is nonplanar, apply Lemma 3.13 in G using these three paths. As G has no K_5 minor, this gives an allowed minor $T(v)$ of B for some $v \in A$. The same argument applies with A_1, A_2 interchanged. □

Lemmas 4.2 and 4.3 give the following test for when an embedding of Ψ_1 extends to one of G.

Corollary 4.14. *Fix $(P_{\mathrm{R}}; s_x, t_y)$ as in Lemma 4.3, where $P_{\mathrm{R}} = P_k$ and $\{i, j, k\} = \{1, 2, 3\}$. A bridge containing P_k must be assigned to H_{rep}, and one containing P_i or P_j to H_{simp}. Every other bridge must have all its attachments in $\{s_i, t_i, s_j, t_j\}$ to be assigned to H_{simp}, or in $\{s_k, t_k, s_x, t_y\}$ to be assigned to H_{rep}. Only the edge s_xt_y, with $x \neq y$, can have both assignments. Then the embedding extends to G exactly when every bridge has an assignment and both test graphs are planar, for at least one assignment of s_xt_y if it is present.*

Proof. Since G is three-connected, every Ψ_1-bridge with nonempty interior has at least three attachments. Also, A_1, A_2 are independent by Lemma 3.4(a). Thus Lemma 4.3 excludes the end faces, leaving only the simple faces and the repeated face. Only the repeated face contains internal vertices of P_k, and only the simple faces contain internal vertices of the other two paths. Hence a bridge containing P_m must be assigned to H_{rep} if $m = k$, and to H_{simp} otherwise. For a bridge outside the paths, its attachments must lie in $\{s_i, t_i, s_j, t_j\}$ to be assigned to H_{simp}, or in $\{s_k, t_k, s_x, t_y\}$ to be assigned to H_{rep}. These sets have only s_x, t_y in common, so a bridge with at least three attachments cannot have both assignments. The only edge that could have both is s_xt_y. If $x = y$, this edge, when present, is P_x itself by (1) and is not a bridge. Thus only s_xt_y with $x \neq y$ can have both assignments. The end faces are unavailable, so a bridge with no assignment cannot be added. Otherwise, Lemma 4.3 requires both test graphs to be planar, with s_xt_y, if present, added to one of them. □

Recall that a completion is obtained by adding edges between the attachments of a bridge. Its relevant completions are those whose planarity is checked when deciding whether H_{simp} or H_{rep} is planar. The next lemma lists these for bridges with three attachments, allowing us to apply Proposition 3.15 to restrict their possible forms.

Lemma 4.15. *Let B be a bridge with attachment set A of size three. Then its triangle completion $B \cup K_3[A]$ is a three-connected minor of G. If this completion is planar, the interior of B is a single vertex. The relevant completions are as follows.*

(i) *If B lies outside the paths, its attachments are c_1, c_2 in one of A_1, A_2 and h in the other. Its relevant completions are*

$$B \cup K_3[A], \qquad B + c_1c_2 + c_1h, \qquad B + c_1c_2 + c_2h.$$

(ii) *If B contains P_m, let e be its third attachment, g the end of P_m in the same one of A_1, A_2 as e, and h the other end. Its relevant completions are*

$$B \cup K_3[A], \qquad B + ge + gh, \qquad B + ge + eh, \qquad B + ge.$$

If the triangle completion is nonplanar and some relevant completion is planar, Proposition 3.15 applies to B, with c_1, c_2, h in case (i), or g, e, h in case (ii), in the roles of x, y, z.

Proof. If B lies outside the paths, its attachment set cannot be A_1 or A_2, since Lemmas 4.2 and 4.5 would give $G \cong \mathfrak{T}_5$. Thus two attachments c_1, c_2 lie in one triple and the third h lies in the other. Lemma 4.13 gives the triangle completion as a three-connected minor of G. If B contains P_m, we may interchange A_1, A_2 and assume its attachments are s_m, t_m, s_x. The paths

$$s_mps_x, \qquad s_xP_xt_xrt_m, \qquad s_mqs_wP_wt_wzt_m,$$

where w is the third index, join the three attachment pairs and have disjoint interiors outside B. Contracting each to an edge gives the triangle completion as a minor of G, and Lemma 3.1 makes it three-connected. In either case, if the triangle completion is planar, Lemma 3.2 makes the interior of B a single vertex.

Now consider a test graph H_{simp} or H_{rep} containing B. Delete the interior and edges of B, retaining A, and call the remaining graph H_1. Consider the bridges on A in H_1. If at least two attach at all three vertices, they and B give a $K_{3,3}$ minor, so the test graph is nonplanar. If exactly one does, Lemma 2.5 requires the triangle completion

of B to be planar. If none does, the condition on B is that $B+E$ is planar, where E contains the edge $\alpha\beta$ whenever a bridge on A in H_1 has attachments $\{\alpha,\beta\}$. All remaining planarity conditions involve only H_1.

Suppose first that B lies outside the paths, and consider the case when no bridge on A in H_1 attaches at all three vertices. In H_{simp}, H_1 contains a cycle through c_1, c_2, h. These three attachments divide the cycle into a c_1–c_2 path, a c_2–h path and an h–c_1 path, each with no attachment in its interior. Each path lies in a bridge whose attachments are its two ends, so c_1c_2, c_2h, c_1h all belong to E. In H_{rep}, H_1 contains a path through c_1, c_2, h or c_2, c_1, h, in that order. In the first case, $c_1c_2, c_2h \in E$, and another bridge can only add c_1h. In the second, $c_1c_2, c_1h \in E$, and another bridge can only add c_2h. Thus the completion is one of the two listed two-edge completions or the triangle completion, giving (i).

If B contains P_m, H_1 contains a path through g, e, h in H_{simp}. When no bridge on A in H_1 attaches at all three, this forces ge, eh into E, giving $B+ge+eh$ or the triangle completion. In H_{rep}, H_1 contains the edge ge, so $ge \in E$. Either, both or neither of gh, eh may also belong to E. This gives precisely the four completions in (ii).

Finally, let B' be an allowed minor of B. Its interior is nonempty and connected, and every attachment still has an interior neighbor. If B contains P_m, choose a g–h path through the interior of B' to replace P_m in Ψ_1. The attachments and the graph outside B are unchanged. For each assignment of the bridges, whether H_{simp} and H_{rep} are planar after the replacement therefore depends only on which of the listed completions of B' are planar. By Lemma 4.3, the same holds for whether the resulting graph is toroidal. Lemma 3.7 now shows that every proper allowed minor of B makes a nonplanar relevant completion planar. Hence Proposition 3.15 applies whenever the triangle completion is nonplanar and some relevant completion is planar. □

The graphs T, S, Q are shown in Figure 4. Forming the candidate graphs in Sections 4.5 and 4.6 also requires a bound on how many bridges can have the same attachments.

Lemma 4.16. *At most three bridges outside the paths have the same attachment set. If a bridge has one interior vertex and three attachments, its attachments are independent. Deleting any bridge outside the paths, with its attachments retained, leaves a graph to which an embedding of Ψ_1 extends.*

Proof. Suppose four bridges outside the paths have the same attachment set A. Each has nonempty interior and at least three attachments. Delete one bridge, retaining A, to obtain a toroidal proper minor of G. In a torus embedding of this minor, Corollary 4.14 forces the other three bridges to be assigned to the same test graph, H_{simp} or H_{rep}. This test graph is planar, but contracting the three bridge interiors and keeping three vertices of A gives a $K_{3,3}$ minor, a contradiction. For the second assertion, the single interior vertex has degree three, so its attachment set is independent by Lemma 3.4(a). For the last assertion, delete any bridge outside the paths while retaining its attachments. This preserves Ψ_1 and gives a proper minor of G, which is toroidal. By Lemma 4.3, the remaining bridges can be assigned to the pair of simple faces and the repeated face so that both H_{simp} and H_{rep} are planar. □

Lemma 4.15 concerns three-attachment bridges with a planar relevant completion. If a bridge has none, the next corollary and minor-minimality shows that it is the *only* bridge and every path outside it is an edge.

Corollary 4.17. *If a bridge B has no planar relevant completion, then $G = \Psi_1 \cup B$ and the paths not in B are edges.*

Proof. If $\Psi_1 \cup B$ were toroidal, Lemma 4.3 would give a planar relevant completion of B, contrary to the hypothesis. Thus $\Psi_1 \cup B$ is nontoroidal, so $G = \Psi_1 \cup B$ by minimality. An internal vertex of a path not contained in B would then have degree two in G, contrary to three-connectivity. Hence every such path is an edge. □

Suppose every bridge with at least two interior vertices has three attachments and a planar relevant completion. The possible bridges are then those with one interior vertex and the forms in Proposition 3.15. At most three four-attachment bridges have the same attachment set, since four would give $K_{4,4}$ and hence a K_5 minor. We combine these bridges with disjoint interiors, subject to the attachment restrictions and Lemma 4.16. A bridge containing P_m includes that path and all Ψ_1-bridges meeting its interior, so its form also accounts for the length of P_m. These enumerations are recorded in [Supp. E3, E4]. The next lemma says that if a bridge with three attachments, together with four vertices having those attachments as their neighborhood, is toroidal, then adding some two edges between its attachments gives a planar graph.

Lemma 4.18. *Let B have three independent attachments x, y, t, with $B - \{x, y, t\}$ nonempty, connected and adjacent to all three. If adding four vertices with neighborhood $\{x, y, t\}$ to B gives a toroidal graph, then at least one of $B + xy + xt$, $B + xy + yt$ and $B + xt + yt$ is planar.*

Proof. Take a torus embedding of the graph obtained by adding the four vertices. These vertices and x, y, t form a nonplanar $K_{3,4}$, so its closed regular neighborhood has genus one. Its faces are therefore disks by Lemma 2.1. With seven vertices and twelve edges, Euler's formula gives five faces. Their boundary walks have total length 24, and each has even length at least four, since $K_{3,4}$ is simple and bipartite and has no vertex of degree one. Thus each walk has length at most $24 - 4 \cdot 4 = 8$. Since $B - \{x, y, t\}$ is connected, it lies in one face. Each attachment has a neighbor in this interior, so the boundary walk contains all three of x, y, t. The walk alternates between these attachments and the added vertices, so it has three or four attachment appearances. Consecutive appearances in cyclic order cannot be equal. Otherwise, the walk would enter and leave an added vertex along the same edge, which is impossible at a vertex of degree three. Thus, after relabeling, the attachment sequence is xyt or $xyxt$. By Lemma 2.2, $B + xy + xt + yt$ is planar for xyt, and $B + xy + xt$ is planar for $xyxt$, using the path yxt. Thus $B + xy + xt$ is planar in either case. □

4.5 When every path P_i is a single edge

This subsection considers the case in which every path is an edge. Here $P_i = s_i t_i$ for each i, so the paths have no internal vertices. We therefore work directly with the Ψ_1-bridges, whose attachments all lie in $A_1 \cup A_2$.

We first classify the graphs with exactly one bridge. By Corollary 4.17, this includes every case in which a bridge has no planar relevant completion. We then show that the remaining bridges with four attachments have one interior vertex, allowing us to form the candidate graphs as described in Section 4.4.

Lemma 4.19. *If every path is an edge and G has exactly one bridge, then G is isomorphic to one of $\mathfrak{T}_5, \mathfrak{T}_{68}, \mathfrak{T}_{69}$.*

Proof. Let B be the unique bridge, with attachment set A. If B were an edge, both ends would lie in $A_1 \cup A_2$, since all edges incident with the twins already belong to Ψ_1. At least one of the three paths, say P_k, would contain neither end. Permuting the path indices in a torus embedding of Ψ_1 gives an embedding with P_k as the repeated path. By Lemma 4.2, each of the two simple faces contains all four ends of the other two paths on its boundary. Both ends of B therefore lie on the boundary of either simple face. Adding B in one of these faces would embed G in the torus, a contradiction. Thus B has nonempty interior and at least three attachments by three-connectivity. If no face in any torus embedding of Ψ_1 contains all of A, Lemma 4.5 gives $G \cong \mathfrak{T}_5$. So suppose such a face exists.

By Lemma 4.2, A has three or four vertices. If it has four, it is either $\{s_i, t_i, s_j, t_j\}$, excluded by Lemma 4.8(a), or $\{s_i, t_i, s_j, t_k\}$ with i, j, k distinct, excluded by Lemma 4.6. Hence $|A| = 3$. A triple containing both s_i, t_i is excluded by Lemma 4.8(b). The remaining triples have three distinct indices and meet both A_1, A_2. Interchange A_1, A_2 if necessary so that A contains two vertices of A_1, then relabel the path indices so that $A = \{s_1, s_2, t_3\}$. By Lemma 4.13, $B \cup K_3[A]$ is a three-connected minor of G.

The only faces containing A are the repeated faces for $(P_1; s_2, t_3)$ and $(P_2; s_1, t_3)$, by Lemma 4.2. Their attachment sequences are s_1, s_2, s_1, t_3 and s_1, s_2, t_3, s_2, respectively. Lemma 2.2 therefore shows that either of the graphs

$$B + s_1 s_2 + s_1 t_3 \quad \text{and} \quad B + s_1 s_2 + s_2 t_3$$

being planar would give a torus embedding of G. Both are therefore nonplanar. Since $B \cup K_3[A]$ contains them, it is also nonplanar.

Contract P_1, P_2 into s_1, s_2 and P_3 into t_3. This leaves B unchanged and gives the four twins p, q, r, z the common neighborhood A. The resulting graph is a proper minor of G, so it is toroidal. Lemma 4.18 therefore makes at least one of the three two-edge completions of B planar. The two containing $s_1 s_2$ are nonplanar, so $B + s_1 t_3 + s_2 t_3$ is planar. The triangle completion is three-connected and nonplanar, and it has no K_5 minor since it is a minor of G. Apply Lemma 3.9 with t_3, s_1, s_2 in the roles of x, y, z. It gives an allowed minor $S(t_3; s_1)$, $S(t_3; s_2)$ or $Q(t_3; s_1, s_2)$, or gives $B = T(s_1)$ or $B = T(s_2)$. The two T cases are impossible, since Lemma 3.10 makes $T(s_1) + s_1 s_2 + s_2 t_3$ and $T(s_2) + s_1 s_2 + s_1 t_3$ planar. By the same lemma, each remaining S or Q minor keeps both completions containing $s_1 s_2$ nonplanar. It also keeps the same attachment set, so replacing B by this minor still gives a nontoroidal graph. Minimality then forces B to be one of these three forms. Exchanging indices $1, 2$ maps the two S forms to each other. Thus either S form gives $G \cong \mathfrak{T}_{68}$, and the Q form gives $G \cong \mathfrak{T}_{69}$ ([Supp. W4]). □

Write $a = s_1$, $b = t_1$, $c = s_2$, $d = t_2$, $e = s_3$, $f = t_3$ and $A = \{a, b, c, d\}$. The boundary of each simple face for P_1, P_2 contains a, b, d, c in this cyclic order.

Lemma 4.20. *Let D be a component of $G - V(\Psi_1)$ with $N(D) = A$. Then no other component of $G - V(\Psi_1)$ has all its attachments in A, and neither ad nor bc is an edge. Let B be the Ψ_1-bridge formed by D and its attaching edges. If B has a disk drawing with a, b, d, c on the boundary in this cyclic order, then D is a single vertex.*

Proof. Suppose another component D' of $G - V(\Psi_1)$ has all its attachments in A. By three-connectivity, it has at least three, so they include a, d or b, c. There is therefore a path joining that pair with internal vertices in D'. Since D is connected and $N(D) = A$, there is a path joining the other pair with internal vertices in D. These paths are disjoint, contrary to Lemma 4.7. An edge ad or bc would give the same contradiction with a path through D joining the other pair.

Now suppose B has the stated disk drawing and D has more than one vertex. Contract D to a vertex v. The resulting proper minor is toroidal, and v lies in a face F of the induced embedding of Ψ_1 whose boundary contains A. By Lemma 4.2, F is a simple face for P_1, P_2 or the repeated face for $(P_1; s_2, t_2)$ or $(P_2; s_1, t_1)$. Its boundary meets $A_1 \cup A_2$ only in A, in one of the cyclic orders

$$a, b, d, c, \qquad a, c, a, b, d, b, \qquad a, c, d, b, d, c,$$

up to reversal. No other bridge with nonempty interior can lie in F, since all its attachments would belong to A, which we have just excluded. Also no edge bridge can lie there. The edges ac, bd are excluded by the independence of A_1, A_2, the edges ad, bc were excluded above, and ab, cd already belong to Ψ_1. Each listed order contains a, b, d, c as a cyclic subsequence. Remove v and its incident edges, and represent F by a disk with distinct boundary appearances kept separate. Choose appearances of a, b, d, c in that order. The Jordan–Schoenflies theorem lets us insert the assumed drawing of B through them, giving a torus embedding of G, a contradiction. □

We now consider a component D of $G - V(\Psi_1)$ with at least four attachments. If $G \ncong \mathfrak{T}_5$, we show that, after relabeling, $N(D) = A$ and its bridge can be drawn in a disk with a, b, d, c in this cyclic order on the boundary. Then Lemma 4.20 implies that D is a single vertex.

Proposition 4.21. *If a component of $G - V(\Psi_1)$ has at least four attachments, then $G \cong \mathfrak{T}_5$, or the component is a single vertex with neighborhood $\{s_i, t_i, s_j, t_j\}$ for some $i \neq j$.*

Proof. Suppose $G \ncong \mathfrak{T}_5$. By Lemmas 4.5 and 4.6, the component has neighborhood $\{s_i, t_i, s_j, t_j\}$ for distinct i, j. Relabel these indices as $1, 2$, so its neighborhood is $A = \{a, b, c, d\}$. Call the component D, and let B be the Ψ_1-bridge formed by D and its attaching edges.

Every path P_i is the edge $s_i t_i$, and the attachments of the Ψ_1-bridges all lie in $A_1 \cup A_2$. By Lemma 4.20, it suffices to show that B has a disk drawing in the order a, b, d, c. For a contradiction suppose it does not. Then $|V(D)| > 1$. By Lemma 4.2, in any torus embedding of a graph containing $\Psi_1 \cup B$, either P_1 is the repeated path and its repeated face also contains s_2, t_2, or P_2 is the repeated path and its repeated face also contains s_1, t_1. In either case B lies in that repeated face.

Suppose there is another component D' of $G - V(\Psi_1)$. If three of its attachments have distinct indices, they either contain a, d or b, c (Lemma 4.7 excludes this), are $\{e, b, d\}$ or $\{f, a, c\}$, or are A_1 or A_2 (Lemma 4.5 excludes this).

For $\{e, b, d\}$, contract D, D' to single vertices. A face containing e, b, d must be the repeated face by Lemma 4.2. $P_3 = ef$ cannot be the repeated path, since it could contain only one additional t-vertex, not both $b = t_1$ and $d = t_2$. Hence the embedding choice is either $(P_1; s_3, t_2)$ or $(P_2; s_3, t_1)$. Neither choice has a face containing all of A, so the contracted vertex representing D cannot be placed in a toroidal embedding, contradicting minor minimality, since $|V(D)| > 1$. Swapping A_1 and A_2 gives the same contradiction for $\{f, a, c\}$.

Therefore the attachments of D' must lie in one of $\{a, b, c, d\}$, $\{a, b, e, f\}$, or $\{c, d, e, f\}$. Lemma 4.20 excludes the first set. Any extra edges outside Ψ_1 between its vertices must join two of a, b, c, d, e, f. Since A_1, A_2 are independent and by Lemma 4.20, only af, eb, cf, ed are possible.

Every remaining attachment triple contains one of ab, cd, ef, so it has two vertices adjacent in G. If its bridge could be drawn in a simple face, its triangle completion would be planar by Lemma 2.2, and Lemma 4.15 would make its interior a single vertex. That vertex has two adjacent neighbors, contradicting Lemma 3.4(a). Thus none of these three-attachment bridges can be drawn in a simple face. D' therefore has either three attachments and a bridge that cannot be drawn in a simple face, or four attachments whose bridge may or may not be drawable in a simple face.

Suppose the bridge B' associated with D' cannot be drawn in a simple face. Since D' has at least three attachments, it cannot be placed in an end face, which contains only two possible attachments. In either choice available to B discussed earlier, the repeated face contains only a, b, c, d among $A_1 \cup A_2$. But D' cannot have all its attachments in $\{a, b, c, d\}$, by Lemma 4.20. Thus $\Psi_1 \cup B \cup B'$ is nontoroidal, and anything else in G can be deleted without making it toroidal.

Suppose instead at least one edge from each of $\{af, eb\}$ and $\{cf, ed\}$ is present in G. No end face can contain any of these edges, as its vertices are in only one of A_1, A_2. In $(P_1; s_2, t_2)$ neither the repeated face nor either simple face can contain an edge from $\{af, eb\}$. In $(P_2; s_1, t_1)$ neither the repeated face nor either simple face can contain an edge from $\{cf, ed\}$.

In either of these two cases, retain $\Psi_1 \cup B$ together with the other bridge, or one edge from each pair, and delete everything else. The resulting graph is still nontoroidal, so minimality makes it equal to G.

Apply Theorem 2.8 to $B \cup C_0$, where $C_0 = abdca$. Its auxiliary graph is three-connected by [17, Lemma 4.2], applied to B in $G + E(C_0)$. Lemma 4.7 excludes crossing paths, so B contains a tripod. Interchanging indices $1, 2$, interchanging A_1, A_2, or doing both lets us take its attachments to be a, b, d. If all attachment paths were trivial, its two trees in D would each be adjacent to a, b, d. Join them through D and contract them to two adjacent vertices. The paths ab, brd and $dzfeqa$ join the three pairs in $\{a, b, d\}$ and have disjoint interiors outside B. Contracting them to edges therefore gives a K_5 minor.

Choose a shortest path from c to a vertex of the tripod in D, with all vertices other than c in D and no earlier vertex on the tripod. Shorten the tripod as in Lemma 3.13. If the path from c meets an attachment path, keep that attachment path nontrivial. Otherwise contract its end into a corner vertex. Shorten the path from c to an edge and contract the remaining interior vertices onto this graph. This gives $T(g)$ on attachments $\{a, b, d\}$, with twins x, y and the remaining edge ug, together with an edge from c to u or one of x, y.

A disk drawing of this reduced bridge with a, b, d, c on the boundary would let us draw the triangle on a, b, d outside the disk. This would make $T(g) + ab + ad + bd$ planar, contrary to Lemma 3.10. The other bridge or edges still exclude both repeated faces, so the minor is nontoroidal. Minimality therefore makes B equal to this reduced bridge.

If $g = d$, a twin has adjacent neighbors a, b, contrary to Lemma 3.4(a). If cx is present, the disjoint paths $auyd, bxc$ for $g = a$, or $ayd, buxc$ for $g = b$, contradict Lemma 4.7. Interchanging x, y excludes cy as well. For cu and $g = b$, use axd, buc. Hence B has edges

$$ua, uc, ux, uy, xb, xd, yb, yd.$$

Delete ua or uc. The remaining bridge has attachments $\{b, c, d\}$ or $\{a, b, d\}$, respectively, and nonplanar triangle completion, so still cannot be drawn in a simple face. The other bridge or edges still exclude the original two embeddings. Since both proper minors are toroidal, Lemma 4.2 forces $(P_2; s_3, t_1)$ for $G - ua$ and $(P_1; s_3, t_2)$ for $G - uc$. The other bridge would have to lie in the repeated face in both embeddings. Its attachments would therefore lie in both $\{b, c, d, e\}$ and $\{a, b, d, e\}$. Three-connectivity then makes its attachment set $\{b, d, e\}$, which was already excluded. In the edge case, the first embedding excludes cf and the second excludes af, leaving eb, ed. Delete y, contract bx, and delete ab, cd and the new bd. This leaves twins p, q with neighborhood $\{a, c, e\}$, twins d, f with neighborhood $\{r, z, e\}$, and edges ua, uc, ub, br, bz, be. This is the proper nontoroidal minor $\mathfrak{T}_6$, a contradiction.

Thus neither of these two cases can occur.

It now remains only to consider D' with exactly four attachments and B' drawable in a simple face. By Lemma 4.20, D' must be a single vertex h. Up to interchanging the indices $1, 2$ of the paths P_i, assume $N(h) = \{a, b, e, f\}$. Contract ha. The edge hb becomes the existing edge ab, hf becomes af, and he becomes ae. Delete ae. The resulting graph is a proper minor and hence toroidal. As B is unchanged, its embedding uses one of $(P_1; s_2, t_2), (P_2; s_1, t_1)$. In the former no face contains both a, f, a contradiction. Thus it uses $(P_2; s_1, t_1)$, and af lies in a simple face for P_1, P_3, whose boundary contains a, b, e, f.

Applying Lemma 4.20 to h excludes another component having all its attachments in $\{a, b, e, f\}$ and excludes af, eb in the original graph. The edges ae, bf are absent by independence, while ab, ef already belong to Ψ_1. Thus af is the only addition inside this face. Erase af, place h in the now empty disk and draw its four edges to a, b, e, f, restoring G without crossings, a contradiction. Therefore there are no extra components D'.

Only the original bridge B and possible extra edges remain outside Ψ_1. There is not an edge from both $\{af, eb\}$ and $\{cf, ed\}$, and Lemma 4.7 says both edges cannot occur in either pair, so there is at most one extra edge. If there are no extra edges, B is the unique bridge. The proper minor G/ef is toroidal, so Lemma 4.8(a), with $H = G$, would imply that G is toroidal. Thus exactly one extra edge remains, and after interchanging indices or exchanging A_1, A_2 or both, assume it is af. Hence

$$G = (\Psi_1 \cup B) + af.$$

We can now decide whether G is toroidal by testing $K = B + ac + cd + bd$ for planarity. Any toroidal embedding of G restricted to Ψ_1 would place B in a face whose boundary contains all of a, b, c, d. If B lies in a simple face of Ψ_1, Lemma 4.3 requires $B + ab + cd + ac + bd \supseteq K$ to be planar. If B lies in the repeated face for $(P_2; s_1, t_1)$, Lemma 4.3 again requires K to be planar. If instead B were placed in the repeated face for $(P_1; s_2, t_2)$, no face could contain both a, f on its boundary. The entire graph G cannot then be drawn.

Conversely, if K is planar, use $(P_2; s_1, t_1)$ and assign B to the repeated face and af to the simple faces for P_1, P_3. The repeated-face test is K, while the simple-face test consists of the cycle $abfea$ and its diagonal af, which is planar. Hence Lemma 4.3 gives a torus embedding. Therefore

$$G \text{ is toroidal} \iff K \text{ is planar.}$$

The same equivalence holds with any allowed minor in place of B. As G is nontoroidal, K is nonplanar.

The graph G has K as a minor by retaining B, the edge cd, and shortening apc, brd to ac, bd. Thus K has no K_5 minor. By the equivalence just proved, replacing B by an allowed minor B' gives a nontoroidal graph whenever $B' + ac + cd + bd$ is nonplanar. Corollary 3.19, with attachments a, b, c, d and added edges ac, cd, bd, therefore lists all 15,626 candidate forms of B. Adjoining Ψ_1 and af to each candidate gives a graph with a K_5 minor or a proper nontoroidal minor [Supp. E2], a contradiction. Therefore B does have a disk drawing in the order a, b, d, c, and Lemma 4.20 implies D is a single vertex. □

We have now determined $\mathfrak{T}_5$ and the graphs with only one bridge. By Corollary 4.17 and Proposition 4.21, every remaining bridge with nonempty interior has either three attachments and a planar relevant completion, or one interior vertex and four attachments. We now bound the number of three-attachment bridges and combine their possible forms with the four-attachment vertices and the edges s_it_j with $i \neq j$.

Theorem 4.22. *A three-connected obstruction containing p, q, r, z with neighborhoods A_1, A_2, where $A_1 \cap A_2 = \varnothing$, and the three edges s_1t_1, s_2t_2, s_3t_3 is isomorphic to one of*

$$\mathfrak{T}_5, \quad \mathfrak{T}_{48}, \quad \mathfrak{T}_{58}, \quad \mathfrak{T}_{68}, \quad \mathfrak{T}_{69}.$$

Proof. Assume $G \ncong \mathfrak{T}_5$. By three-connectivity, Lemma 4.5 and Proposition 4.21, every bridge B with nonempty interior has three or four attachments. Each three-attachment set meets both A_1, A_2. Every four-attachment bridge has one interior vertex attached to $\{s_i, t_i, s_j, t_j\}$ for some $i \neq j$. If some three-attachment bridge has no planar relevant completion, Corollary 4.17 makes it the only bridge, and Lemma 4.19 implies $G \cong \mathfrak{T}_{68}$ or $\mathfrak{T}_{69}$. Thus assume every three-attachment bridge has a planar relevant completion.

Consider H, a toroidal subgraph obtained from G by deleting entire bridges while retaining their attachments and Ψ_1. Take a torus embedding of H. By Lemma 4.3, both H_{simp} and H_{rep} are planar. Suppose for a contradiction that a three-attachment bridge B is assigned to H_{simp}. Its attachment set A consists of three vertices of $C = s_it_it_js_js_i$, so it contains both ends of either P_i or P_j. If its interior were a single vertex, it would have degree three in G with two adjacent neighbors, the two path ends, contradicting Lemma 3.4(a). Therefore Lemma 4.15 makes its triangle completion nonplanar. But $H_{\text{simp}} \supseteq B \cup C$, and contracting an edge of C incident with its unique vertex outside A gives that triangle completion as a minor of H_{simp}, contradicting planarity. Therefore all three-attachment bridges of H are assigned to H_{rep}, whose four attachments are s_k, t_k, s_x, t_y.

Write m for the number of these bridges. Contract the interior of each to one vertex, keeping only these m vertices, the four attachments, and one edge from each contracted vertex to each of its three attachments. This forms a simple planar bipartite graph with $m + 4$ vertices and $3m$ edges. Thus

$$3m \leq 2(m + 4) - 4, \qquad m \leq 4.$$

Deleting one three-attachment bridge of G, if one exists, leaves a toroidal proper minor with at most four remaining such bridges. Thus G itself has at most five. Suppose it has five. Then there is no four-attachment bridge or edge s_it_j with $i \neq j$, since deleting it would leave all five in a toroidal proper minor.

By Lemma 4.15 and Proposition 3.15, a three-attachment bridge on $\{c_1, c_2, h\}$, with $c_1, c_2 \in A_1$ and $h \in A_2$, up to exchanging A_1, A_2, has one interior vertex or is one of

$$T(c_1), \quad T(c_2), \quad T(h), \quad S(c_1; c_2), \quad S(c_2; c_1).$$

If the attachment triple contains an edge P_i, label it c_1h. Both $T(c_2)$ and $S(c_1; c_2)$ have a degree-three vertex adjacent to both c_1, h, so Lemma 3.4(a) excludes them. The single-vertex bridge is excluded for the same reason. This leaves $T(c_1), T(h), S(c_2; c_1)$.

These give 72 labeled forms. Among the eighteen triples meeting both A_1, A_2, twelve contain an edge P_i, from three choices of that edge and four choices of the third vertex. The other six contain none, giving $12 \cdot 3 + 6 \cdot 6 = 72$. Form graphs by adding these bridges to Ψ_1 with disjoint interiors, using at most three on any one triple by Lemma 4.16 and at most five in total. By Lemma 4.20, there is at most one four-attachment vertex on each set $\{s_i, t_i, s_j, t_j\}$. We also allow any subset of the six edges s_it_j with $i \neq j$. We retain only graphs satisfying Lemma 3.4(a) and Lemma 4.20, and exclude disjoint s_i–t_j and s_j–t_i paths through distinct bridges by Lemma 4.7. The obstruction check in [Supp. E3] leaves only $\mathfrak{T}_{48}$ and $\mathfrak{T}_{58}$. Neither graph has a four-attachment bridge. Together with $\mathfrak{T}_5$, $\mathfrak{T}_{68}$ and $\mathfrak{T}_{69}$, these give the graphs in the statement. □

4.6 Three-attachment bridges and four-attachment bridges with one interior vertex

For the remainder of Section 4, we suppose now that some path P_i is not an edge. For this subsection, we consider the case where every bridge with four attachments has one interior vertex. By Proposition 4.12, every bridge with at least two interior vertices then has three attachments. The next corollary of Corollary 4.17 and Lemma 4.8 shows that these bridges have planar relevant completions.

Corollary 4.23. *Every bridge with three attachments has a planar relevant completion.*

Proof. A bridge with one interior vertex has a planar triangle completion. Suppose that a bridge B has no planar relevant completion. By Corollary 4.17, $G = \Psi_1 \cup B$ and the paths outside B are edges. Since some path is not an edge, B contains a path P_m. Its third attachment belongs to $\{s_x, t_x\}$ for some $x \neq m$. Every Ψ_1-bridge in B has an attachment in the interior of P_m and all its attachments in $V(P_m) \cup \{s_x, t_x\}$. Let w be the third index. The graph $G/s_w t_w$ is a toroidal proper minor, so Lemma 4.8(c), after relabeling the indices, makes G toroidal. □

Proposition 4.24. *If some P_i is not an edge and every bridge with four attachments has one interior vertex, then G is isomorphic to one of*

$$\mathfrak{T}_{32}, \quad \mathfrak{T}_{38}, \quad \mathfrak{T}_{43}, \quad \mathfrak{T}_{45}, \quad \mathfrak{T}_{48}, \quad \mathfrak{T}_{61}, \quad \mathfrak{T}_{66}, \quad \mathfrak{T}_{67}.$$

Proof. By Corollary 4.23, every three-attachment bridge has a planar relevant completion. Lemma 4.15 and Proposition 3.15 therefore restrict its form. We use these forms and the attachment sets from Proposition 4.12 to construct a finite collection containing G.

Begin with $J_1 \cup J_2$, and allow any subset of the nine edges between A_1 and A_2. On each of the eighteen triples $\{c_1, c_2, h\}$, with c_1, c_2 in one of A_1, A_2 and h in the other, a bridge may have one interior vertex or one of the forms

$$T(c_1), \quad T(c_2), \quad T(h), \quad S(c_1; c_2), \quad S(c_2; c_1).$$

A bridge containing P_m may also have the form $Q(h; g, e)$ or either four-cycle form of Lemma 3.14, with g, e, h as in Lemma 4.15 and in the roles of x, y, z. These additional forms occur on the twelve triples containing some s_m, t_m, giving $18 \cdot 6 + 12 \cdot 3 = 144$ labeled three-attachment forms.

Every four-attachment bridge has one interior vertex. Its neighborhood is one of the three sets $\{s_i, t_i, s_j, t_j\}$ or the six type (c) sets $\{s_m, t_m, s_x, t_y\}$ with $\{m, x, y\} = \{1, 2, 3\}$. Together with the three-attachment forms and the nine possible edges, this gives $144 + 9 + 9 = 162$ labeled edge and bridge types. We combine them on $A_1 \cup A_2$, keeping their interiors disjoint.

Lemma 4.16 allows at most three bridges outside the paths on any one triple, counting all forms together. If the triple contains s_m, t_m and $s_m t_m$ is absent, allow one further bridge for the possible bridge containing P_m. On each set $\{s_i, t_i, s_j, t_j\}$, allow at most three single vertices, since four give $K_{4,4}$ and hence a K_5 minor. A Q, four-cycle or type (c) bridge must contain its associated path P_m. Thus, for each m, at most one bridge among these forms can occur, and it requires $s_m t_m$ to be absent.

Exclude a graph if a degree-three interior vertex has adjacent neighbors, or if both $s_i t_j$ and $s_j t_i$ occur for $i \neq j$, by Lemmas 3.4(a) and 4.7. If $s_m t_m$ is absent, require a bridge with both s_m, t_m among its attachments, since P_m lies in such a bridge. We do not choose the paths inside the candidates or require three disjoint s_m–t_m paths. The family therefore includes additional graphs, but every graph under consideration occurs in it. Since some P_m is not an edge, (1) gives $s_m t_m \notin E(G)$. We therefore consider candidates with at least one of these three edges absent. Among those whose bridges all have at most one interior vertex, the enumeration [Supp. E4] leaves $\mathfrak{T}_{32}$, $\mathfrak{T}_{43}$, $\mathfrak{T}_{61}$. Among those with a bridge of form T, S or Q, it leaves $\mathfrak{T}_{38}$, $\mathfrak{T}_{45}$, $\mathfrak{T}_{48}$, $\mathfrak{T}_{66}$, $\mathfrak{T}_{67}$.

No candidate with a four-cycle form remains. These checks cover all the bridge forms, with graphs containing both a T, S, Q form and a four-cycle form included in both checks. Every other candidate has a torus embedding, a K_5 minor, a cut of order at most two or a proper nontoroidal minor, as recorded in [Supp. E4]. The same family with all three edges $s_m t_m$ present gives $\mathfrak{T}_{48}$ and $\mathfrak{T}_{58}$, already covered by Theorem 4.22. The three checks above therefore give the stated list. □

4.7 A bridge with four attachments and at least two interior vertices

We now consider a bridge with four attachments and at least two interior vertices. We treat types (c), (a) and (b) of Proposition 4.12 in that order.

Proposition 4.25. *If G has a bridge of type (c) with at least two interior vertices, then $G \cong \mathfrak{T}_{21}$ or $G \cong \mathfrak{T}_{63}$.*

Proof. Let B be such a bridge. Number the paths so that B contains P_1 and its other attachments are s_2, t_3. Exchange d and f in the notation of Section 4.5. The ends of P_1, P_2, P_3 are now (a, b), (c, f) and (e, d), respectively, and the attachment set of B is $A = \{a, b, c, d\}$. The bridge B consists of P_1 and all Ψ_1-bridges meeting its interior. We first determine the embedding choice that could extend to G. If P_2 were repeated, the interior of P_1 would occur only on the boundaries of the simple faces following P_1, P_3. Neither contains c, so the bridge joining c to the interior of P_1 could not be drawn. Similarly, if P_3 were repeated, the simple faces following P_1, P_2 would miss d. Therefore P_1 must be repeated, and the bridges joining its interior to c, d require both vertices on the repeated-face boundary. By Lemma 4.2, the choice must be $(P_1; s_2, t_3)$. Figure 11 shows this embedding.

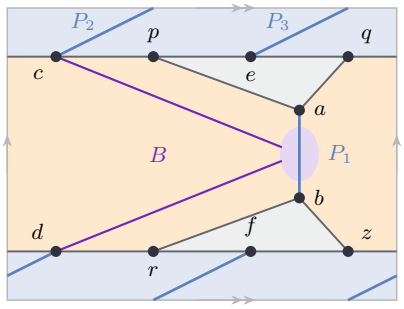


Figure 11: The torus embedding of Ψ_1 for the choice $(P_1; s_2, t_3)$, with B in the repeated face.

For this choice, H_{rep} contains $B + ac + bd$, while H_{simp} contains P_2, P_3 and the added edges ce, fd. The boundaries of the repeated face and the simple faces share only c, d among the vertices where a bridge can attach. Since a bridge with nonempty interior has at least three attachments, only the possible edge cd can be assigned to either test. Leave it out of both for now.

Contract the interior of B to a single vertex w. This gives a proper minor, since B has at least two interior vertices, and hence a toroidal graph. The path $a - w - b$ replaces P_1, and the edges wc, wd still force the same embedding choice by the preceding argument. All remaining bridges can therefore be drawn in their available faces. The simple-face test is unchanged by the contraction, so H_{simp} is planar. Also, another bridge of type (c) containing P_2 or P_3 would attach at a or b. It would have to lie in a simple face, but neither simple face contains a or b. Thus no such bridge occurs.

Suppose for a contradiction that $cd \in E(G)$. The proper minor $G - cd$ is toroidal, so both H_{simp} and H_{rep} are planar. If $H_{\text{simp}} + cd$ were planar, Lemma 4.3 would give a torus embedding of G. Hence $H_{\text{simp}} + cd$ must be nonplanar.

If a bridge joins $x \in V(P_2) - \{c\}$ to $y \in V(P_3) - \{d\}$, it contains an x–y path with internal vertices outside Ψ_1. This path is disjoint from cd. Its end x follows c on P_2, whereas y precedes d on P_3, contrary to Lemma 4.7. Therefore no bridge joins $P_2 - c$ to $P_3 - d$. Each bridge in H_{simp} has all its attachments in one of $V(P_2) \cup \{d\}$ and $V(P_3) \cup \{c\}$.

Let L consist of P_2, the added edge fd, and all bridges with attachments in $V(P_2) \cup \{d\}$. Let R consist of P_3, the added edge ce, and all bridges with attachments in $V(P_3) \cup \{c\}$. Then $H_{\text{simp}} = L \cup R$, and L, R meet only at c, d. The graph L contains the path $cP_2f\,d$. Starting with H_{simp}, retain R and this path and delete everything else belonging to L. Contract all but one edge of the path to obtain $R + cd$. Thus $R + cd$ is a minor of the planar graph H_{simp}, so it is planar. Interchanging L, R shows that $L + cd$ is planar as well. Gluing their plane drawings along cd gives a plane drawing of $H_{\text{simp}} + cd$. Since H_{rep} is also planar, assigning cd to the simple faces gives a torus embedding of G by Lemma 4.3, a contradiction. Hence $cd \notin E(G)$.

Since H_{simp} is planar, H_{rep} must be nonplanar. No additional bridge can be assigned to the simple faces. Otherwise, deleting that bridge while retaining its attachments would give a proper minor with B and the nonplanar repeated-face test unchanged. The same embedding choice is still forced, so this minor would remain nontoroidal, contradicting minimality. Further, neither P_2 nor P_3 can have an internal vertex. Contracting an edge of either path while keeping its ends distinct would again give a proper minor with the same forced choice and the same nonplanar repeated-face test. Therefore

$$P_2 = cf, \qquad P_3 = ed, \qquad H_{\text{simp}} = cfdec.$$

Suppose another bridge D with nonempty interior attaches at both a, b. It contains an a–b path Q through its interior. Replace P_1 by Q to form a new subdivision $\Psi_1' = J_1 \cup J_2 \cup Q \cup P_2 \cup P_3$. Relative to Ψ_1', the connected interior of B lies outside the subdivision and attaches at all of a, b, c, d. These are s_1, s_2, t_1, t_3, contradicting Lemma 4.6. Every other bridge D with nonempty interior lies in the repeated-face test. It has at least three attachments among a, b, c, d and cannot attach at both a, b, so its attachment set is $\{a, c, d\}$ or $\{b, c, d\}$.

We now show that each such D has exactly one interior vertex and that at most one bridge of each attachment type occurs. Contract the interior of B to w again. The resulting proper minor is toroidal with the same forced embedding choice, so its repeated-face test is planar. The subgraph consisting of D, w, and the three edges from w to the attachments of D is therefore planar. Relative to that attachment triple it has exactly two bridges, D and the star centered at w. Lemma 2.5(ii) makes the triangle completion of D planar, and Lemma 3.2 makes its interior a single vertex. Finally, two such vertices with the same attachment triple, together with w, would give three vertices adjacent to the same three attachments. This gives a $K_{3,3}$ in the planar repeated-face test. Thus there is at most one vertex h_a with neighborhood $\{a, c, d\}$ and at most one vertex h_b with neighborhood $\{b, c, d\}$.

We next determine which edges between a, b, c, d can occur. The edges ac, bd are absent because A_1, A_2 are independent. The edge ab is absent by (1), since P_1 is nontrivial, and we have already excluded cd. Thus only ad, bc are possible. If h_a is present, ad must be absent by Lemma 3.4(a). Likewise, if h_b is present, bc is absent. Up to simultaneously interchanging a, b and c, d, the possible additions outside B are therefore

$$\varnothing, \qquad \{ad\}, \qquad \{h_a\}, \qquad \{ad, bc\}, \qquad \{ad, h_b\}, \qquad \{h_a, h_b\}.$$

Each displayed vertex includes its three incident edges.

Fix one of these six configurations. Let H_{ext} consist of A together with its additional vertices and edges. The repeated-face test is then

$$H_1 = (B \cup H_{\text{ext}}) + ac + bd.$$

The graph H_1 is nonplanar. To obtain it as a minor of G, retain B, H_{ext}, and the outside paths apc, brd, delete everything else, and shorten these paths to ac, bd. Thus H_1 has no K_5 minor and contains a $K_{3,3}$ subdivision H_2.

Let B_0 be obtained from B by one allowed operation. Choose an a–b path through its connected interior to replace P_1. The connections to c, d still force $(P_1; s_2, t_3)$, and replacing B by B_0 gives a toroidal proper minor. Hence

$$(B_0 \cup H_{\text{ext}}) + ac + bd \quad \text{is planar.}$$

Put $H_0 = H_{\text{ext}} + ac + bd$. For each of the six configurations, H_0 is planar, $V(B) \cap V(H_0) = A$, and every vertex of $H_0 - A$ has all its neighbors in A. We have shown that $B_0 \cup H_0$ is planar after every single allowed operation on B. Lemma 3.17, applied to B, A, H_0 and H_2, therefore gives $B' = B$. Write H_2' for the resulting subdivision. Thus B has the form in that lemma and has at most $|A| + 5 = 9$ interior vertices.

Deleting any present edge among

$$ad, \quad bc, \quad h_ac, \quad h_ad, \quad h_bc, \quad h_bd$$

gives a toroidal proper minor while leaving B and the forced embedding choice unchanged. If an edge at h_a or h_b is deleted, the remaining two-edge path has ends a, c, a, d, b, c or b, d. A path with ends a, d or b, c must still lie in the repeated face. A path with ends a, c or b, d can be drawn alongside the added edge ac or bd in the planar repeated-face test, even if it was placed in an end face. Thus deleting the same edge from H_1 makes it planar. Every $K_{3,3}$ subdivision of H_1, including H_2', must therefore use every listed edge that is present. In particular, it contains every present h_a, h_b. When h_a occurs, deleting h_ac and contracting ah_a gives the planar graph $H_1 - h_a + ad$. Similarly, when h_b occurs, deleting h_bd and contracting bh_b shows that $H_1 - h_b + bc$ is planar.

We establish the final restrictions before enumerating the possible forms. Suppose neither h_b nor bc occurs. Contract cf and take a torus embedding of this proper minor. Identify the two boundary copies of P_1 in its incident facial disks, keeping all other boundary appearances distinct. The path P_1 now lies inside the resulting region apart from its ends. The rotation calculation [Supp. R3] gives three possible regions: an annulus with attachment sequence $ac \mid bd$, or a disk with attachment sequence

$$a, c, b, d, c \qquad \text{or} \qquad a, c, b, c, d, c.$$

Every Ψ_1-bridge inside B meets the interior of P_1, so lies in this region. No other face contains both a, d, so any present h_a or ad lies there as well. Thus the region contains $B \cup H_{\text{ext}}$.

The annulus would make H_1 planar by Lemma 2.4. The first disk would make $H_1 + bc + cd$ planar by Lemma 2.3, again making H_1 planar. Both are impossible. The remaining disk permits the added edges ac, bc, cd, so $(B \cup H_{\text{ext}}) + ac + bc + cd$ is planar. This graph contains $H_1 - bd$, which is therefore planar. Every $K_{3,3}$ subdivision of H_1, including H_2', must then use bd. Symmetrically, when neither h_a nor ad occurs, contracting ed shows that $H_1 - ac$ is planar, so every such subdivision must use ac.

Finally, apply Construction 3.18 with the six choices of H_{ext}, the added edges ac, bd, and the required subdivision edges and planar completions established above. This lists every possible B. Restoring the twin subgraphs $J_1 \cup J_2$, the edges cf, ed, and the chosen exterior configuration gives the 1,586 candidates for G. The checks in [Supp. E5] leave only $G \cong \mathfrak{T}_{21}$ or $G \cong \mathfrak{T}_{63}$. □

It remains to consider types (a) and (b). Let B_a be a bridge of type (a) with attachment set $\{s_x, t_x, s_y, t_y\}$, or let B_b be a bridge of type (b) containing P_m, with additional attachments s_x, t_x. In either case, assume the bridge has at least two interior vertices. Let w be the index different from x, y for B_a, or from m, x for B_b. For B_a, put

$$C = s_x t_x t_y s_y s_x, \qquad C' = s_x t_y t_x s_y s_x.$$

For B_b, write $a = s_m$, $b = t_m$, $c = s_x$ and $d = t_x$, so its attachment set is $\{a, b, c, d\}$. For either bridge B with attachment set A, the graph $B \cup K_4[A]$ is three-connected. After deleting at most two vertices, the remaining attachments lie in one component because they form a clique. Any other component would consist only of interior vertices of B and would also be separated in G by the same deletion, contrary to three-connectivity.

The next lemma lets us contract a bridge of type (a) or (b) to a tree and then put the bridge back.

Lemma 4.26. *Let B be B_a or B_b, with attachment set A, and let T be an allowed minor of B that is a tree whose vertices of degree one are exactly the four attachments. Consider a torus embedding of the graph obtained from G by replacing B by T. A thin closed disk around T meets the rest of the embedding precisely in A. We call the cyclic order of the attachments on its boundary the order around T.*

(i) *If T consists of four edges at a vertex h, the order around T is that of C or C' for B_a, and of $acdba$ or $acbda$ for B_b.*

(ii) *If the order around T is that of a four-cycle C_0 on A for which $B \cup C_0$ is planar, then G is toroidal.*

Proof. For (i), the order around T is the cyclic order of the four edges around h. Consider first B_a. These edges lie in one face of Ψ_1 whose boundary contains all four attachments. By Lemma 4.2, it is a simple face following P_x, P_y, whose boundary walk passes s_x, t_x, t_y, s_y, or the repeated face for $(P_x; s_y, t_y)$, whose boundary walk passes $s_x, t_x, t_y, t_x, s_x, s_y$, or the same with x, y interchanged. Whichever appearances of s_x and t_x the edges use, among the four appearances used s_y is next to s_x and t_y is next to t_x. Thus s_x, s_y are consecutive around h, and so are t_x, t_y, which gives the order of C or C'.

For B_b, the path ahb replaces P_m, and hc, hd join its internal vertex to s_x, t_x. By Lemma 4.2, hc, hd lie in the simple faces following P_m, P_x, whose boundary walks pass a, h, b, d, c, or in the repeated face for $(P_m; s_x, t_x)$, whose boundary walk passes a, h, b, d, b, h, a, c. In both cases c is next to a and d is next to b on the boundary walk. So when hc, hd leave h on the same side of the path ahb, the edge hc lies between ha and hd, and the order is a, c, d, b. When they leave on different sides, the order is a, c, b, d.

For (ii), Lemma 2.2 gives a disk drawing of B meeting the boundary precisely in A, with the attachments in the cyclic order of C_0. Replacing the drawing of T inside the thin closed disk by this drawing, using the Jordan–Schoenflies theorem, gives a torus embedding of G. □

Lemma 4.27. *The graphs $B_a \cup C$, $B_a \cup C'$, $B_b \cup acdba$ and $B_b \cup acbda$ are nonplanar.*

Proof. Let B be either B_a or B_b. Contract its interior to a vertex h, and let T consist of the four edges at h. This gives a proper minor of G, since B has at least two interior vertices, so it has a torus embedding. By Lemma 4.26(i), the order around T is that of one of the two listed four-cycles. If both four-cycle completions were planar, Lemma 4.26(ii) would make G toroidal. Thus at least one of them is nonplanar.

It remains to consider the case where exactly one four-cycle completion is planar. For B_a, suppose first that $B_a \cup C$ is planar. Then B_a has a disk drawing with attachment order s_y, s_x, t_x, t_y, but none with order s_y, s_x, t_y, t_x, since the latter would make $B_a \cup C'$ planar. Lemma 2.9 therefore gives disjoint s_y–t_y and s_x–t_x paths, with internal vertices in the interior of B_a. If instead $B_a \cup C'$ is planar, the same lemma gives disjoint s_y–t_x and s_x–t_y paths. Both have their internal vertices outside Ψ_1, contradicting Lemma 4.7. Thus this second case is excluded.

For B_b, apply Lemma 2.9 to the orders a, c, d, b and a, c, b, d. If $B_b \cup acdba$ is planar, we obtain disjoint a–b and c–d paths. If $B_b \cup acbda$ is planar, we obtain disjoint a–d and c–b paths. Again, the internal vertices lie in the interior of B_b.

In each case that remains, we have two disjoint paths through B, each joining A_1 to A_2, whose four ends are the attachments. Each path has a nonempty interior, since B has no edges between its attachments. Start with the two sets of internal vertices of these paths. Repeatedly add an unassigned interior vertex to a set containing one of its neighbors. Since the interior of B is connected, this extends the two sets to disjoint connected sets covering the interior of B, with at least one edge between them.

Contract these sets to vertices u, v. Retain the edge uv and the two edges from each of u, v to the ends of its original path, and delete all other edges of the resulting bridge. Call this bridge B'. Its interior is nonempty and connected, its four labeled attachments are unchanged, and each attachment has an interior neighbor. These contractions and deletions therefore give an allowed minor of B.

Write C_0, C_1 for the two four-cycles under consideration, with $B \cup C_0$ nonplanar and $B \cup C_1$ planar. The ends of the two chosen paths alternate on C_0, so $B' \cup C_0$ is $K_{3,3}$. One vertex class consists of u and the two attachments adjacent to v, and the other consists of v and the two attachments adjacent to u. Suppose $B' \neq B$. Replacing B by B' gives a proper minor of G, so it has a torus embedding. The bridge B' is a tree whose vertices of degree one are the four attachments. The four-cycle joining the attachments in the order around B' can be drawn along the boundary of a thin closed disk around B', so adding it to B' gives a planar graph. Since $B' \cup C_0$ is nonplanar, this order cannot be that of C_0. Contracting uv preserves the order, so Lemma 4.26(i) leaves only the order of C_1. Lemma 4.26(ii), with $T = B'$, then makes G toroidal, a contradiction. Hence $B = B'$.

It follows that u, v have degree three in G. By Lemma 3.4(a), the two attachments adjacent to u are nonadjacent, and the same holds for those adjacent to v. Use the two paths of length two through u, v, together with the unchanged P_w, as new connecting paths. When $B_b \cup acbda$ is the planar completion, interchange t_m, t_x so that these paths again join corresponding indices.

The ends of each new path of length two are nonadjacent, so both satisfy (1) and the unchanged P_w still satisfies it as well. The three paths are disjoint and meet $A_1 \cup A_2$ only at their ends. The remark after Lemma 4.11 therefore allows us to apply that lemma to the new subdivision. But uv is a bridge joining the interiors of two of its paths, a contradiction. Hence neither four-cycle completion can be planar, as required. □

Proposition 4.28. *If some P_i is not an edge, then no bridge of type (a) has at least two interior vertices.*

Proof. Suppose B is such a bridge, with attachment set $A = \{s_x, t_x, s_y, t_y\}$. We first show that at least one of

$$B \cup s_y s_x t_x t_y, \qquad B \cup s_x s_y t_y t_x$$

is planar. Suppose both are nonplanar. In a torus embedding of $\Psi_1 \cup B$, Lemma 4.2 would place B in a simple face following P_x, P_y, or in the repeated face for $(P_x; s_y, t_y)$ or $(P_y; s_x, t_x)$. The corresponding test graph of Lemma 4.3 contains B together with $P_x, P_y, s_x s_y, t_x t_y$ in the first case, $P_x, s_x s_y, t_x t_y$ in the second and $P_y, s_x s_y, t_x t_y$ in the third. Each contains a subdivision of one of the two path completions, so it is nonplanar. Thus $\Psi_1 \cup B$ is nontoroidal, and minimality gives $G = \Psi_1 \cup B$. Three-connectivity then makes every path an edge, contrary to the hypothesis. Interchange x, y if necessary so that $B \cup s_y s_x t_x t_y$ is planar.

We next restrict the bridges outside B. A path through another Ψ_1-bridge from s_x to t_y, or from s_x to an internal vertex of P_y, would be disjoint from a t_x–s_y path through the interior of B. These paths join P_x, P_y in opposite order, contrary to Lemma 4.7. Interchanging x, y, interchanging A_1, A_2, or doing both gives the corresponding exclusions. Consequently,

$$s_x t_y, s_y t_x \notin E(G), \qquad W_x, W_y \subseteq \{s_w, t_w\}.$$

Every other bridge outside the paths with nonempty interior has at least three attachments. Any three vertices of A include the pair s_x, t_y or the pair s_y, t_x, so no such bridge has all its attachments in A. Lemma 4.7 also excludes disjoint s_x–t_y and s_y–t_x paths through B itself.

Consider the embedding choice $(P_x; s_y, t_y)$. For this choice to extend, the following three conditions outside B are necessary:

(1) P_x is an edge;

(2) every other $(J_1 \cup J_2)$-bridge, apart from the single-edge paths, attaches within $\{s_y, t_y, s_w, t_w\}$;

(3) the simple-face test H_{simp} is planar.

To see the first condition, if P_x were nontrivial, three-connectivity would give $W_x \neq \varnothing$. Since $W_x \subseteq \{s_w, t_w\}$, some Ψ_1-bridge would join the interior of P_x to s_w or t_w. It would have to lie in the repeated face, which contains neither vertex. When P_x is an edge, the preceding exclusions leave no bridge other than B that can use the repeated face or an end face. Every other bridge must therefore use the simple faces, whose attachments in $A_1 \cup A_2$ lie in $\{s_y, t_y, s_w, t_w\}$. This gives the second and third conditions. Conversely, if all three hold, the repeated-face test is $B \cup s_y s_x t_x t_y$, which is planar, and the simple-face test is planar by (3). Lemma 4.3 then gives a torus embedding of G. The same description applies with any allowed minor of B in its place.

Since G is nontoroidal, at least one of these three conditions fails. We claim that they cannot all hold with x, y interchanged either. Suppose they do. Then P_y is an edge, and $B \cup s_x s_y t_y t_x$ must be nonplanar, since otherwise $(P_y; s_x, t_x)$ would extend to G. Choose a nontrivial path P_k, where $k \in \{x, w\}$, and let j be the other index in $\{x, w\}$. Three-connectivity gives the bridge containing P_k an attachment other than s_k, t_k. The second condition, with x, y interchanged, makes this attachment s_j or t_j. In the first case, contract the connected interior of this bridge into t_k and in the second, contract it into s_k. Shorten the remaining paths to edges and retain only the resulting Ψ_1, B, and the new edge. This edge is $s_x t_w$ or $s_w t_x$, and the minor is proper because P_k was nontrivial.

This minor is still nontoroidal. If $P_R = P_w$, the simple-face test contains the nonplanar graph $B \cup C$. For $(P_y; s_x, t_x)$, the repeated-face test is the nonplanar graph $B \cup s_x s_y t_y t_x$. For $(P_x; s_y, t_y)$, no face contains both ends of the new edge. By Lemma 4.2, these are all the possibilities for a face containing the four attachments of B. This contradicts minimality. Thus at least one of the three conditions fails for each of $(P_x; s_y, t_y)$ and $(P_y; s_x, t_x)$.

Now let B_0 be obtained from B by any one allowed operation. Replacing B by B_0 gives a proper minor of G, so it is toroidal. The four attachments of B are retained. Also, the path lengths, the attachments of the other bridges, and the simple-face tests for the preceding two choices are unchanged. Thus neither choice satisfies all three conditions, and any torus embedding of this minor must have $P_R = P_w$.

This forces P_x, P_y to be edges. Otherwise, three-connectivity of G and $W_x, W_y \subseteq \{s_w, t_w\}$ give an unchanged Ψ_1-bridge joining the interior of a nontrivial one to s_w or t_w. With P_w repeated, this bridge must lie in a simple face, but neither simple face contains these vertices. Since some path is nontrivial, P_w is therefore nontrivial. The simple-face test contains $B_0 \cup C$, so

$$B_0 \cup C \text{ is planar for every } B_0 \text{ obtained by one allowed operation.}$$

The paths $P_x, P_y, s_x p s_y, t_x r t_y$ give $B \cup C$ as a minor of G, so it has no K_5 minor. It is nonplanar by Lemma 4.27, and every $K_{3,3}$ subdivision in it uses $s_y t_y$, since deleting this edge leaves the planar graph $B \cup s_y s_x t_x t_y$. Corollary 3.19, with $H_0 = C$, therefore lists the 1,378 possible forms of B, with $s_y t_y$ required in the subdivision. Up to interchanging A_1, A_2, the enumeration in [Supp. E6] leaves only one form for which $B \cup K_4[A]$ is three-connected, there are no disjoint s_x–t_y and s_y–t_x paths, and no degree-three interior vertex has adjacent neighbors in G. Its interior is $\{p', q', v\}$, where

$$N(p') = N(q') = \{s_x, s_y, v\},$$

and its remaining edges are vt_x, vt_y.

It remains to exclude this form. Since P_w is nontrivial, three-connectivity gives $W_w \neq \varnothing$. The form of B is unchanged by interchanging x, y, so assume W_w contains s_y or t_y. Delete vt_x. The remaining bridge has attachments $\{s_x, s_y, t_y\}$. In $(B - vt_x) \cup C$, contract $t_x t_y$ to a vertex t and delete $s_x s_y$. The resulting graph is $K_{3,3}$ with parts $\{p', q', t\}$ and $\{s_x, s_y, v\}$. Thus $(B - vt_x) \cup C$ is nonplanar.

We check every embedding choice that could contain the remaining bridge. If $P_R = P_w$, it must use the simple faces, since no other face contains both s_x, s_y. Their test contains $(B - vt_x) \cup C$, so it is nonplanar. For $(P_x; s_y, t_y)$ and $(P_y; s_x, t_x)$, the bridge still requires the repeated face. For each choice at least one of conditions (1)–(3) still fails outside B. By Lemma 4.2, the only additional possibility is $(P_y; s_x, t_w)$. However, an unchanged Ψ_1-bridge joins the interior of P_w to s_y or t_y. For this choice, the only faces meeting that interior are the simple faces following P_x, P_w, and neither contains s_y or t_y. Consequently $G - vt_x$ is a proper nontoroidal minor of G, contrary to minimality. □

It remains to consider type (b). Let $B = B_b$, and put $e = s_w$ and $f = t_w$.

Lemma 4.29. *The path P_x is the edge cd, and up to interchanging A_1, A_2, the bridge B is one of the five bridges $B_1, \ldots, B_5$ below. Moreover, one of the following holds.*

(i) *$P_w = ef$, and $G = (\Psi_1 \cup B) + eb$ or $G = (\Psi_1 \cup B) + af$.*

(ii) *Every other $(J_1 \cup J_2)$-bridge attaches within $\{c, d, e, f\}$, and H_{simp} for $(P_m; s_x, t_x)$ is nonplanar.*

In each row below, p', q' have the displayed common neighborhood, and the last column gives all remaining edges. The interior vertices are p', q', u, v, except that B_5 has no vertex u.

	$N(p') = N(q')$	*remaining edges*
B_1	$\{a, c, v\}$	$au,\ bu,\ du,\ uv$
B_2	$\{a, c, v\}$	$au,\ bu,\ dv,\ uv$
B_3	$\{a, u, v\}$	$cu,\ bv,\ dv$
B_4	$\{d, u, v\}$	$cu,\ av,\ bv$
B_5	$\{a, b, v\}$	$cv,\ dv$

Proof. Put $A = \{a, b, c, d\}$. We first restrict the bridges outside B. Since the interior of B is connected, there is a path from each of c, d to the interior of P_m, with internal vertices outside Ψ_1. Suppose a Ψ_1-bridge outside B contains a path from b to c or to an internal vertex of P_x. This path is disjoint from the path in B from d to the interior of P_m. On P_m, the end of the path from d precedes b, while on P_x the end of the path from b precedes d. The two paths therefore contradict Lemma 4.7. Interchanging A_1, A_2 also excludes a path outside B from a to d or to the interior of P_x. Thus ad, bc are absent and $W_x \subseteq \{e, f\}$. Every bridge outside the paths with nonempty interior has at least three attachments. Any three vertices of A include a, d or b, c, so no such bridge attaches only in A.

We next show that $B + ac + bd$ is planar. Suppose not. In a torus embedding of $\Psi_1 \cup B$, Lemma 4.2 would place the bridges joining c, d to the interior of P_m in the simple faces following P_m, P_x or in the repeated face for $(P_m; s_x, t_x)$. In both cases the test graph of Lemma 4.3 containing P_m contains $B + ac + bd$, so it is nonplanar. Thus $\Psi_1 \cup B$ is nontoroidal, so minimality and three-connectivity give $G = \Psi_1 \cup B$, with $P_x = cd$ and $P_w = ef$. The proper minor G/ef is toroidal. Applying Lemma 4.8(c), with P_m, P_x, P_w in the roles of P_1, P_2, P_3, would make G toroidal as well, a contradiction.

Consider the possible repeated paths. If $P_{\mathrm{R}} = P_x$, the bridges joining c, d to the interior of P_m must lie in the simple faces for P_m, P_w. Neither simple face contains c or d, so this is impossible. If $P_{\mathrm{R}} = P_m$, these bridges require both $c = s_x$ and $d = t_x$ on the repeated face. Thus the embedding choice must be $(P_m; s_x, t_x)$. The exclusions above prevent any Ψ_1-bridge outside B from using this repeated face, so its test is exactly the planar graph $B + ac + bd$. Since G is nontoroidal, the obstruction to this choice lies outside B. Therefore either another bridge has no face containing its attachments, or H_{simp} is nonplanar. Replacing B by an allowed minor changes neither of these conditions. It also retains paths from c, d to a chosen a–b path through the new interior, so $P_{\mathrm{R}} = P_x$ remains impossible. Any torus embedding after such a replacement must therefore have $P_{\mathrm{R}} = P_w$.

Let B_0 be obtained from B by one allowed operation, and replace P_m by an a–b path through the interior of B_0. This gives a proper minor of G, so it has a torus embedding, with repeated path P_w. We claim that $P_x = cd$. Otherwise, three-connectivity of G and $W_x \subseteq \{e, f\}$ give a Ψ_1-bridge joining the interior of P_x to e or f. It is unchanged by the operation on B and must use a simple face. Neither simple face contains e or f, a contradiction.

The simple faces now follow the chosen a–b path and the edge cd. The first paragraph excludes every Ψ_1-bridge outside B_0 from these faces, so their test is $B_0 + ac + cd + bd$. Thus this graph is planar for every single allowed operation on B. Restoring B changes only this simple-face test and the other bridge assignments and the planar repeated-face test remain unchanged. Consequently,

$$K = B + ac + cd + bd$$

is nonplanar, since otherwise Lemma 4.3 would give a torus embedding of G.

We now distinguish whether another $(J_1 \cup J_2)$-bridge attaches at a or b. Suppose first that a bridge outside the paths attaches at a. If it does not attach at f, then its attachment set must be $\{a, b, e\}$. Indeed, the first paragraph excludes d and the pair b, c, and Lemma 4.5 excludes $\{a, c, e\} = A_1$. A bridge with nonempty interior has at least three attachments, while the possible single edges ab, ac, ae are absent. Thus the bridge contains either an a–f path or an e–b path. Interchanging A_1, A_2 gives the same conclusion for a bridge attaching at b. Contract this path to the edge af or eb, and shorten P_w to ef. If instead the bridge containing P_w attaches at a or b, contract its interior into f or e, respectively, to obtain the same kind of edge.

Retain only B, the resulting Ψ_1, and this new edge. The resulting minor is nontoroidal. The repeated path cannot be P_x, as above. If it is P_m, the additional endpoints must be s_x, t_x, and no face contains both ends of af or eb. If it is P_w, the simple-face test is the nonplanar graph K. These exhaust the repeated paths. Minimality therefore forces this minor to be all of G, giving $P_w = ef$ and $G = (\Psi_1 \cup B) + af$ or $G = (\Psi_1 \cup B) + eb$. This proves (i).

If no other $(J_1 \cup J_2)$-bridge attaches at a or b, all its attachments lie in $\{c, d, e, f\}$. For the choice $(P_m; s_x, t_x)$, every Ψ_1-bridge outside B can then be assigned to the simple faces. Since the repeated-face test $B + ac + bd$ is planar and G is nontoroidal, the simple-face test must be nonplanar. This gives (ii).

It remains to determine B. Shortening the paths apc, brd to ac, bd and retaining $P_x = cd$ gives K as a minor of G, so K has no K_5 minor. It is nonplanar, and every $K_{3,3}$ subdivision in it uses cd, since $K - cd = B + ac + bd$ is planar. Let H_0 be the path $a - c - d - b$. The graph $B_0 \cup H_0$ is planar after every single allowed operation, as shown above. Corollary 3.19 now lists the 2,104 possible forms of B, with cd required in the retained subdivision. The enumeration [Supp. E7] keeps those for which $B + ac + bd$ is planar, every single allowed operation makes $B + ac + cd + bd$ planar, $B \cup K_4[A]$ is three-connected, and neither K nor the minor $\Psi_1 \cup B$ with P_w shortened to ef has a K_5 minor. Nine labeled bridges remain. Up to interchanging A_1, A_2, these are the five displayed forms. □

Proposition 4.30. *If G has a bridge of type (b) with at least two interior vertices, then $G \cong \mathfrak{T}_{66}$, or the twins and paths can be chosen as at the start of this section so that one of the following holds.*

(i) *All three paths are edges.*

(ii) *There is a bridge of type (c) with at least two interior vertices.*

(iii) *Every bridge with nonempty interior has three attachments.*

Proof. Let B be such a bridge. After interchanging A_1, A_2 if necessary, Lemma 4.29 gives $P_x = cd$ and makes B one of $B_1, \ldots, B_5$. In case (i) of that lemma, the check in [Supp. E7] gives $G \cong \mathfrak{T}_{66}$. In case (ii), every other $(J_1 \cup J_2)$-bridge attaches within $\{c, d, e, f\}$, and the simple-face test for $(P_m; s_x, t_x)$ is nonplanar. In particular,

af, eb are absent. Write H_{simp} for this test. It consists of P_w, cd, the other bridges, and the added edges ce, df. We consider the five possible forms of B.

Suppose first that $B = B_1$ or B_2. In the proper subgraph $G - ap$, use p', q' and r, z as the twins, with triples $\{a, c, v\}$ and $\{f, d, b\}$, and paths

$$P_1' = aqeP_w f, \qquad P_2' = cd, \qquad P_3' = vub.$$

We show that no torus embedding of this subgraph is possible. Apply Lemma 4.2 to the new subdivision. If its repeated path is P_1', then u lies only on the simple faces, which miss a. Thus the edge au cannot be drawn. If its repeated path is P_2', then q lies only on the simple faces, which miss c, so cq cannot be drawn. The repeated path must therefore be P_3'.

Now u lies only on the repeated face. The edge au forces a onto this face, and in B_1 the edge du also forces d onto it. In B_2, the edge dv gives the same requirement as the only other face containing v is an end face that misses d. Thus the new embedding choice is $(P_3'; a, d)$.

Each original Ψ_1-bridge outside B is still a bridge of the new subdivision, with attachments in $V(P_w) \cup \{c, d\}$. The repeated face meets this set only at d, and the two end faces meet it only at c and f, respectively. Each such bridge has at least two attachments, so it must lie in the simple faces. The path cpe must also lie in a simple face, since e is internal to P_1'. Consequently, the new simple-face test contains a subdivision of the original H_{simp} and it retains P_w, the edge cd, and all the other original bridges. The path cpe replaces ce, and df is one of its added edges. This test is nonplanar. Lemma 4.3 therefore makes $G - ap$ a proper nontoroidal subgraph, contrary to minimality.

Suppose next that $B = B_3$. Use p', q' and r, z as the twins, with triples $\{a, u, v\}$ and $\{f, d, b\}$, and paths

$$apeP_w f, \qquad ucd, \qquad vb.$$

These paths are disjoint and meet the new triples only at their ends. They also satisfy (1), as the first two have nonadjacent ends because af, ud are absent, and the third is an edge. The path eqc has ends internal to the first two paths and its internal vertex q is outside the new subdivision. Its bridge therefore meets the interiors of two of the new paths, contrary to Lemma 4.11 and the remark after its proof.

Suppose $B = B_5$. Let H_B consist of $B - dv$, the paths apc, brd, and the edge cd. This is a $K_{3,3}$ subdivision with corner sets $\{p', q', c\}$ and $\{a, b, v\}$. The six edges from p', q' to a, b, v are in B, and the three paths from c to a, b, v are $cpa, cdrb, cv$. In particular, $H_B - \{c, d\}$ is connected, and d has degree two in H_B.

Replace the added edges ce, df in H_{simp} by the paths cqe, dzf, and call the resulting graph H'_{simp}. It is connected and nonplanar, since it is a subdivision of H_{simp}. The graphs H_B and H'_{simp} meet exactly in the edge cd. Their union is a proper subgraph of G, since it omits dv. Suppose this union has a torus embedding. By Youngs's theorem, the restriction to the connected nonplanar graph H'_{simp} is cellular. The connected graph $H_B - \{c, d\}$ is disjoint from it, so lies in one disk face. Thus $H_B - cd$ can be drawn in a disk representing that face, with its attachments at the boundary appearances of c, d.

Keep the used appearances of c as separate boundary vertices. Only one appearance of d is needed, since d has only one incident edge in $H_B - cd$. Glue a second disk along the first disk's boundary, giving a sphere. In the second disk, draw a vertex c^* joined to every boundary copy of c and to the single copy of d. Contract all edges from c^* to copies of c. These contractions identify the copies of c to one vertex, and the remaining edge from c^* to d supplies the missing edge cd. We have obtained a planar drawing of H_B, contradicting that it is a $K_{3,3}$ subdivision. Hence the union is a proper nontoroidal subgraph of G, again contradicting minimality.

The only remaining form is $B = B_4$. Use p, q and p', q' as the twins, with triples $\{a, c, e\}$ and $\{v, u, d\}$, and choose connecting paths with minimum total length. Outside the four chosen twins, a has only the neighbor v, and u has only the neighbor c. Since each connecting path meets the triples only at its ends, the paths at these vertices must be av and cu. The third path joins e to d. If ed is an edge, minimality of the total length makes this third path ed, giving (i).

Suppose ed is absent. Delete the six vertices of the new triples and the four chosen twins. Let D_0 be the component containing b, r, z, f and the interior of the old P_w. It has neighbors d, e, v, and possibly c. Every other component has all its neighbors in $\{c, d, e\}$, since outside the deleted vertices, v has only the neighbor b, while a, u have no neighbors. The interior of the new e–d path lies entirely in one of these components.

If $N_G(D_0) = \{c, d, e, v\}$, that path must run through D_0. Otherwise D_0 is a connected set outside the new subdivision, attached at all four of c, d, e, v. The vertices e, d are the ends of one connecting path, and c, v are ends of the other two, contrary to Lemma 4.6. Thus the bridge with interior D_0 contains the new e–d path and has additional attachments c, v on the other two paths. It is a bridge of type (c), with at least two interior vertices since it contains b, r, z, f. This gives (ii).

If $N_G(D_0) = \{d, e, v\}$, every other component has its neighbors in $\{c, d, e\}$ and must have at least three by three-connectivity. Hence every component left by the deletion has exactly three neighbors in the new triples. These components are precisely the nonempty interiors of the bridges for the new twin subgraphs, so every such bridge has three attachments. This gives (iii). □

Proof of Theorem 4.1. We may assume $G \not\cong \mathfrak{T}_4$ and $G \not\cong \mathfrak{T}_5$, since both are in the stated list. By Proposition 4.12, every bridge with nonempty interior has three or four attachments. If all three paths are edges, Theorem 4.22 gives the result. Suppose now that some path is not an edge. If every bridge with four attachments has one interior vertex, Proposition 4.24 applies. Otherwise, let B have four attachments and at least two interior vertices. Type (a) is excluded by Proposition 4.28, and type (c) gives $\mathfrak{T}_{21}$ or $\mathfrak{T}_{63}$ by Proposition 4.25. For type (b), Proposition 4.30 gives $\mathfrak{T}_{66}$, or new twins and paths satisfying (i), (ii) or (iii). If all three new paths are edges, Theorem 4.22 applies. Otherwise, (ii) gives a bridge of type (c) with at least two interior vertices, so Proposition 4.25 applies, and (iii) leaves no bridge with four attachments, so Proposition 4.24 applies. Every graph obtained belongs to the list. □

5 Two twin triples with two common vertices

In this section, we consider the case where an obstruction has no two disjoint twin triples but has some that share two vertices. Let the two triples be $A_1 = \{a, b, c\}$ and $A_2 = \{a, b, d\}$, where a, b, c, d are distinct, and call them the c triple and the d triple. Choose twins p, q with neighborhood A_1 and r, z with neighborhood A_2, and write J_1, J_2 for their twin subgraphs. Put $\Psi_2 = J_1 \cup J_2$. Figure 12 shows the twin triples and the graph Ψ_2. We determine G from the bridges of Ψ_2.

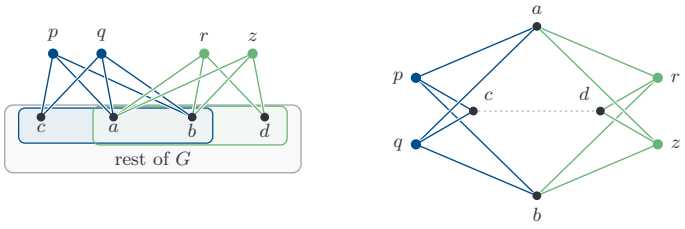


Figure 12: Left, two twin triples meeting at a, b. Right, the graph Ψ_2, with the possible edge cd dotted.

Only cd can join a, b, c, d, by Lemma 3.4(a). Moreover, every twin has degree three, so each component of $G - V(\Psi_2)$ has its neighbors among a, b, c, d. It has at least three of them, as G is three-connected. The Ψ_2-bridges are these components with their attaching edges, and the edge cd when present. Since $\Psi_2 = J_1 \cup J_2$, these are also exactly the bridges on $A_1 \cup A_2$ in $G - \{p, q, r, z\}$. Here the two bridge descriptions used in Section 4 coincide.

Theorem 5.1. *A graph is a three-connected obstruction in which some two twin triples share two vertices, and no two twin triples are disjoint, if and only if it is isomorphic to one of*

$$\mathfrak{T}_2, \mathfrak{T}_9, \mathfrak{T}_{10}, \mathfrak{T}_{11}, \mathfrak{T}_{15}, \mathfrak{T}_{16}, \mathfrak{T}_{17}, \mathfrak{T}_{18}, \mathfrak{T}_{19}, \mathfrak{T}_{22}, \mathfrak{T}_{24}, \mathfrak{T}_{26},$$
$$\mathfrak{T}_{30}, \mathfrak{T}_{34}, \mathfrak{T}_{39}, \mathfrak{T}_{40}, \mathfrak{T}_{51}, \mathfrak{T}_{52}, \mathfrak{T}_{53}, \mathfrak{T}_{56}, \mathfrak{T}_{57}, \mathfrak{T}_{72}, \mathfrak{T}_{73}, \mathfrak{T}_{74}.$$

We divide the proof according to whether some choice of the two twin pairs gives a Ψ_2-bridge attached to all four vertices a, b, c, d, or no choice does. We first establish the embedding tests and reductions needed in both cases. Sections 5.3 and 5.4 complete the first case, and Sections 5.5 and 5.6 finish the second.

Lemma 5.2. *Suppose B is a Ψ_2-bridge with attachments a, b, c, d. Then B is the only such bridge, cd is absent, and every other Ψ_2-bridge has a single interior vertex with neighborhood $\{a, b, c\}$ or $\{a, b, d\}$. If at least four vertices have the same neighborhood $\{a, b, c\}$ or $\{a, b, d\}$, then $G \cong \mathfrak{T}_2$.*

Proof. Suppose another Ψ_2-bridge meets both c, d, or cd is an edge. Then there is a c–d path P whose internal vertices avoid Ψ_2 and the interior of B. Contracting the interior of B to a vertex and shortening P to cd gives a K_5 minor [Supp. W5], a contradiction. Thus B is the only bridge with all four attachments, cd is absent, and every other component of $G - V(\Psi_2)$ has neighborhood $\{a, b, c\}$ or $\{a, b, d\}$ by three-connectivity.

Suppose such a component D with neighborhood $\{a, b, c\}$ has nonplanar triangle completion, and write B_D for its bridge. Its completion is three-connected by Lemma 3.1. The paths arb, apc, bqc join the three attachment pairs with disjoint interiors outside B_D, so Lemma 3.13 gives $T(t)$ as an allowed minor of B_D for some $t \in \{a, b, c\}$. Contracting the unique edge of $T(t)$ incident with t gives two further vertices with neighborhood $\{a, b, c\}$. Contracting the interior of B to a vertex then gives a proper $\mathfrak{T}_2$ minor [Supp. W6], contrary to minimality.

Thus every such triangle completion is planar. The same argument applies to $\{a, b, d\}$ after interchanging c, d and the two twin pairs. Lemma 3.2 makes each component a single vertex. If at least four vertices have either neighborhood, contracting the interior of B and retaining four vertices with that neighborhood and two with the other gives a $\mathfrak{T}_2$ minor [Supp. W7]. Minimality therefore gives $G \cong \mathfrak{T}_2$. □

If B exists and $G \ncong \mathfrak{T}_2$, then Lemma 5.2 shows that G consists of B, n_c vertices whose neighborhood is the c triple and n_d whose neighborhood is the d triple. These counts include the chosen twin vertices, so each is at least two, and the lemma bounds each by three. Thus (n_c, n_d) is $(2, 2)$, $(3, 2)$ or $(3, 3)$, after interchanging c, d if needed.

5.1 Faces and embedding tests of Ψ_2

The next lemma gives the embedding tests for these three cases.

Lemma 5.3. *Let B have four independent attachments a, b, c, d, with $B - \{a, b, c, d\}$ nonempty, connected and adjacent to all four. For $(n_c, n_d) \in \{(2, 2), (3, 2), (3, 3)\}$, the graph obtained from B by adding n_c vertices with neighborhood $\{a, b, c\}$ and n_d with neighborhood $\{a, b, d\}$ is toroidal if and only if a graph in the row of (n_c, n_d) is planar:*

(n_c, n_d)	*planarity tests*
$(3, 3)$	$B + ac + bc + bd + ad$
$(3, 2)$	$B + ab + ac + bc + ad$, $\quad B + ab + ac + bc + bd$
$(2, 2)$	$B + ac + bd$, $\quad B + ad + bc$,
	$B + ab + ac + ad$, $\quad B + ab + bc + bd$.

Proof. Let G be the graph in the statement, and let K consist of a, b, c, d, the $n_c + n_d$ twins, and their incident edges. Then $G = B \cup K$ and $V(B) \cap V(K) = \{a, b, c, d\}$. In particular, $K = \Psi_2$ when $(n_c, n_d) = (2, 2)$.

Suppose G has a torus embedding and restrict it to K. The connected interior of B lies in one face F, whose boundary contains all four attachments. For each of the three pairs (n_c, n_d), the rotations of K and their boundary attachment sequences are given in [Supp. R4]. Let $\mathcal{N}$ be a closed regular neighborhood of K. By Lemma 2.1, if $\mathcal{N}$ has genus one, every complementary region is a disk. If it has genus zero, the complementary regions are disks together with exactly one annulus or one torus with an open disk removed. For a disk face, let H_0 consist of the attachments and the edges joining consecutive attachments on its boundary, as in the proof of Lemma 2.2.

First suppose $(n_c, n_d) = (2, 2)$ and $\mathcal{N}$ has genus zero. No individual boundary walk meets all four attachments, so F cannot be a disk or a torus with an open disk removed. It must therefore be an annulus. The possible boundary pairs meeting all four attachments are $ac \mid bd$ and $ad \mid bc$, giving the tests

$$B + ac + bd, \qquad B + ad + bc$$

by Lemma 2.4. If $\mathcal{N}$ instead has genus one, F is a disk. The seven boundary sequences meeting all four attachments give seven tests by Lemma 2.2. Five contain either the pair ac, bd or the pair ad, bc, so their planarity implies one of the annular tests. The remaining two are

$$B + ab + ac + ad, \qquad B + ab + bc + bd.$$

For $(n_c, n_d) = (3, 3)$, K is nonplanar, so $\mathcal{N}$ has genus one and F is a disk. The only boundary attachment sequence meeting all four attachments is $acbd$, giving the test

$$B + ac + bc + bd + ad$$

by Lemma 2.2. For $(n_c, n_d) = (3, 2)$, K is again nonplanar. The boundary sequences meeting all four attachments similarly give

$$B + ab + ac + bc + ad, \qquad B + ab + ac + bc + bd, \qquad B + ac + bc + bd + ad.$$

The third test is redundant. In a sphere embedding of this graph, the cycle $acbda$ bounds two disks. The connected interior of B lies in one of them, so we can add ab in the other. Deleting bd or ad then gives either of the first two tests. Conversely, suppose one of the graphs in the table is planar. Choose an embedding of K and a face that gave this entry. For a disk face, Lemma 2.2 makes planarity of $B \cup H_0$ equivalent to a drawing of B in that face. For either annular face, Lemma 2.4 gives the corresponding drawing. Inserting it into the chosen face of K embeds $G = B \cup K$ on the torus. □

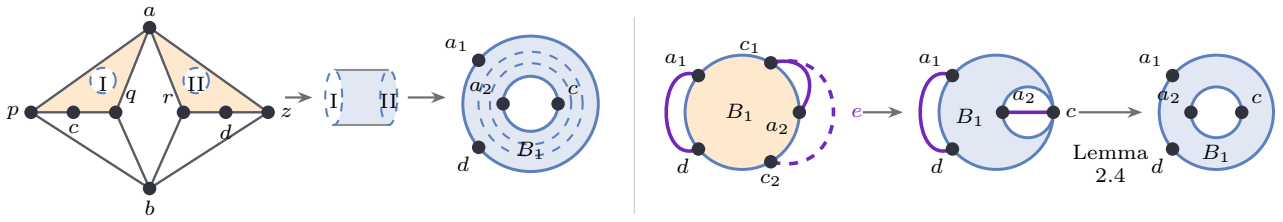
Lemma 5.3 gives the planarity conditions for G to embed in the torus. We now show that the completions used in these tests are minors of G, so they also have no K_5 minor.

Lemma 5.4. *Let B be a Ψ_2-bridge of G with attachments a, b, c, d, and let E be a set of edges among a, b, c, d other than cd. Then $B + E$ is a K_5-minor-free minor of G. If the edges of E among any three of a, b, c, d connect them, then $B + E$ is three-connected.*

Proof. Contract one twin from each of the neighborhoods $\{a, b, c\}, \{a, b, d\}$ into a and the other into b. These contractions give all edges among a, b, c, d except cd. Keeping B and the edges of E gives $B+E$ as a minor of G, so it has no K_5 minor. For three-connectivity, delete at most two vertices from $B + E$. Every remaining component contains a surviving attachment, since otherwise the deleted vertices separate that component from the rest of G. If at most one attachment is deleted, the surviving edges of E connect all remaining attachments, so the remaining graph is connected. If two attachments are deleted, no interior vertex is deleted. The interior of B therefore remains connected, and both surviving attachments still have neighbors in it. Thus the remaining graph is connected. □

Lemma 5.5. *No three-connected obstruction has twin pairs with neighborhoods $\{a, b, c\}, \{a, b, d\}$ and $\deg(b) = 4$.*

Proof. Suppose $\deg(b) = 4$. Every component outside Ψ_2 then has neighborhood $\{a, c, d\}$ by three-connectivity. Let B_1 consist of their bridges and cd, if present, so $G = \Psi_2 \cup B_1$. By minimality, $G - z$ has a torus embedding. All bridges in B_1 lie in the same face F of $H = \Psi_2 - z$, since each meets d and $\deg_H(d) = 1$. Record only a, c, d on the boundary walks of F. If consecutive appearances are equal, move their incident edges to one appearance along the intervening boundary arc, preserving their order. Since B_1 meets both c, d, the rotation table [Supp. R5] and Lemma 2.1 leave only the disk sequences $acd, acad, acacd$ and the annular pairs $ad \mid ac, ad \mid c$. We show that B_1 can be drawn in an annulus with a_1, d on one boundary and a_2, c on the other, allowing the edges at a to use a_1 or a_2. The annular cases already give this, with a_2 unused for $ad \mid c$. In a disk case, separate the appearances and use the boundary order a_1, c_1, a_2, c_2, d, leaving positions unused where needed. Draw c_1c_2, c_1a_2, a_1d outside the disk without crossings, then contract c_1c_2 to recover c. The resulting graph is planar, with a_1, a_2 distinct and a_1d, a_2c added. Lemma 2.4 gives the required drawing (Figure 13(d)–(f)). A planar embedding of Ψ_2 has faces $apcqa$ and $ardza$. Remove an open disk from each and join the new boundary circles by a cylinder. This gives a torus embedding with an annular face $ac \mid ad$. Insert the drawing of B_1, with the edges at a_2 reaching a between ap, aq and those at a_1 reaching a between ar, az. This embeds G on the torus (Figure 13(a)–(c)), a contradiction. □



(a) Remove the two disks | (b) Join them by a cylinder | (c) Insert B_1 $a_1, a_2 \mapsto a$ | (d) Add e, c_1a_2, a_1d | (e) Contract e and recover c | (f) $a_2c \mid a_1d$

Figure 13: The proof of Lemma 5.5. A torus embedding of G in (a)–(c) and an annular drawing of B_1 in (d)–(f).

Moreover, we can exclude the case of $(n_c, n_d) = (3, 3)$ before reducing the bridges.

Lemma 5.6. *If G has a Ψ_2-bridge B with attachments a, b, c, d, then $(n_c, n_d) \neq (3, 3)$.*

Proof. Suppose $(n_c, n_d) = (3, 3)$, and put $C = acbda$ and $H = B \cup C$. By Lemmas 5.4 and 5.3, H is three-connected and nonplanar. The graph $\mathrm{Aux}(H, C)$ is also three-connected, since the added vertex has four neighbors. Thus outcome (c) of Theorem 2.8 is impossible, so $H - E(C) = B$ contains crossing paths or a tripod.

Suppose first that B contains crossing paths P from a to b and Q from c to d. Both have nonempty interiors in the interior of B. Choose a shortest path R in the interior of B joining their interiors. Let $W_1 = V(Q)$ and let W_2 consist of the internal vertices of P and the vertices of R outside W_1. Then W_1, W_2 are disjoint and connected. Each of $\{a\}, \{b\}, W_1$ is adjacent to W_2 and to each of the six vertices with neighborhoods $\{a, b, c\}, \{a, b, d\}$, giving a $K_{3,7}$ minor. Since both path interiors are nonempty, $|V(G)| \geq 12 > 10$, so this minor is proper, contradicting minimality.

Now suppose B contains a tripod on $A \subseteq \{a, b, c, d\}$. Lemma 3.12 gives a minor H_A consisting of a, b, c, d, the six vertices with neighborhoods $\{a, b, c\}, \{a, b, d\}$, and two vertices with neighborhood A. This is $\mathfrak{T}_{13}$ if $a, b \in A$ and $\mathfrak{T}_9$ otherwise [Supp. W8]. If $|\mathrm{Int}(B)| > 2$, then H_A has fewer vertices than G. Otherwise the connected interior consists of two adjacent vertices, which are nonadjacent in H_A. Thus H_A is a proper minor in both cases, contradicting minimality. □

5.2 Reducing bridges while preserving the tests

The later proofs reduce bridges with three attachments obtained by separating B at a three-cut. A reduction must preserve the conditions in the embedding test, so that an embedding of the reduced graph would give one of the original graph. Proposition 5.7 lists the bridges that cannot be reduced without changing which of their eight completions are planar, provided one completion by two edges is planar.

Proposition 5.7. *Let B be a bridge of G on $A = \{x, y, t\}$ whose triangle completion is three-connected and a minor of G, and suppose $B + xy + xt$ is planar. Assume that every proper allowed minor B' has $B' + E$ planar and $B + E$ nonplanar for some $E \subseteq E(K_3[A])$. Then B consists of one interior vertex adjacent to x, y, t, or is one of $T(y), T(t), Q(x; y, t), S(x; y), S(x; t)$.*

Proof. By Proposition 3.11, $B - x$ is a chain of edges and four-cycles. If there is no four-cycle, B has one interior vertex. With exactly one four-cycle, the paths from y, t to the cycle have lengths at most two. Lengths $(0, 0)$ disconnect the interior, while $(0, 1), (1, 0), (1, 1)$ give $T(t), T(y), Q(x; y, t)$. Otherwise, there is one four-cycle with an end path of length two, or at least two four-cycles. After interchanging y, t if necessary, contract the end path at y into y, leaving the first cycle as $ypuqy$. If this is the only cycle, let f be the middle vertex of the length-two end path at t. Otherwise, let f be one of the two vertices of the next cycle other than its connecting vertices. Proposition 3.11(ii),(iv) gives $xf \in E(B)$. Keep a u–t path through f, the cycle $ypuqy$, and the edges xp, xq, xf. Delete the other interior vertices, starting with those furthest from the interior vertices of this path and cycle. Keep only this path, the cycle and the edges xp, xq, xf. Contract the internal vertices between u, f into u and those between f, t into t. This keeps the interior connected and a neighbor at each attachment, giving the allowed minor $S(x; y)$. By Lemma 3.10, every completion of this minor containing yt is nonplanar, so the corresponding completion of B is also nonplanar. Every completion of B not containing yt is a subgraph of the planar graph $B + xy + xt$. Thus the minor has exactly the same planar completions as B and cannot be proper by the hypothesis. Hence $B = S(x; y)$, or $B = S(x; t)$. □

5.3 A bridge with four attachments: nonplanar B or $(n_c, n_d) = (3, 2)$

We now classify the graphs for which some choice of the twin pairs gives a Ψ_2-bridge B attached to all four vertices a, b, c, d. Apart from $\mathfrak{T}_2$, Lemmas 5.2 and 5.6 show that $(n_c, n_d) = (2, 2)$ or $(3, 2)$.

Lemma 5.8. *If G has a Ψ_2-bridge B with attachments a, b, c, d and $(n_c, n_d) = (3, 2)$, $B + ab + ac + bc$ is nonplanar.*

Proof. Suppose $H = B + ab + ac + bc$ is planar. By Lemma 5.3, both $H + ad$ and $H + bd$ are nonplanar. Since H is connected and contains ab, Lemma 2.7 shows that

$$(H + ad)/ab = (H + bd)/ab = (H + ad + bd)/ab$$

is nonplanar, where parallel edges are identified. By Lemma 5.4, this is a K_5-minor-free minor of G, so it contains a $K_{3,3}$ subdivision. Write x for the vertex obtained by identifying a, b. Apply Lemma 3.17 to $(B + ab)/ab$, with the fixed planar graph on x, c, d having edges xc, xd. Only a segment with ends c, d can retain an internal vertex. There is at most one such segment, so the reduced interior has at most six corner vertices and one other vertex.

All contractions are within the interior, so carry them out in B while keeping a, b distinct. Keep the edges selected by the reduction, choosing one edge av or bv for each edge xv of the reduced bridge. If a or b has no interior neighbor, keep one additional edge at that vertex. Such an edge is available because every original interior vertex survives or is contracted into a retained interior vertex. Delete the remaining edges to obtain an allowed minor B' with at most seven interior vertices. Both tests for B' contain the retained subdivision after contracting ab. By Lemma 5.3, replacing B by B' therefore gives a nontoroidal minor of G, so minimality gives $B' = B$.

Construction 3.18, with attachments x, c, d and added edges xc, xd, lists the reduced bridges before x is separated. Replace each edge xv by av or bv and add an edge to the interior at a or b if needed, considering every possible choice. This includes every reduction above. Adjoin three vertices with neighborhood $\{a, b, c\}$ and two with neighborhood $\{a, b, d\}$. The enumeration [Supp. E9] gives 2,953 candidates up to isomorphisms fixing the attachments. Of these, 2,159 have a K_5 minor, 92 have a cut of at most two vertices, and 702 have a proper nontoroidal minor. These exclude every candidate, so H cannot be planar. □

Proposition 5.9 (Nonplanar completions with four attachments)**.** *If B is nonplanar and $G = B \cup \Psi_2$, then G is isomorphic to one of $\mathfrak{T}_{17}, \mathfrak{T}_{18}, \mathfrak{T}_{19}, \mathfrak{T}_{22}, \mathfrak{T}_{24}, \mathfrak{T}_{26}, \mathfrak{T}_{30}, \mathfrak{T}_{34}, \mathfrak{T}_{39}, \mathfrak{T}_{48}, \mathfrak{T}_{52}, \mathfrak{T}_{57}, \mathfrak{T}_{66}, \mathfrak{T}_{74}$. If $(n_c, n_d) = (3, 2)$, $G \cong \mathfrak{T}_{11}$.*

The last graph in this list, $\mathfrak{T}_{74}$, is the one missing from the collection of Myrvold and Woodcock [23].

Proof. Take $E = \varnothing$ in the first case and $E = \{ab, ac, bc\}$ in the second. In either case, $B + E$ is nonplanar by the hypothesis or Lemma 5.8, and is contained in every applicable test of Lemma 5.3. By Lemma 5.4, it is a K_5-minor-free minor of G. Every applicable test for an allowed minor B' of B contains $B' + E$. By Lemma 5.3, replacing B by B' therefore gives a nontoroidal graph whenever $B' + E$ is nonplanar. Corollary 3.19, with attachments a, b, c, d and added edges E, lists every possible B. Adjoin two vertices with each of the neighborhoods $\{a, b, c\}$ and $\{a, b, d\}$ in the first case, and three and two, respectively, in the second. This gives 187,075 graphs. The records in [Supp. E8] leave exactly the stated graphs. Every other graph is excluded by five branch sets of a K_5 minor, a cut of at most two vertices, or an edge whose deletion or contraction leaves a nontoroidal graph. These contradict, respectively, K_5-minor-freeness, three-connectivity or minimality. □

5.4 A bridge with four attachments: planar B and $(n_c, n_d) = (2, 2)$

It remains to consider $G = B \cup \Psi_2$ with B planar and $(n_c, n_d) = (2, 2)$. By Lemma 5.3, all four graphs

$$B + ac + bd, \qquad B + ad + bc, \qquad B + ab + ac + ad, \qquad B + ab + bc + bd$$

are nonplanar. Let H_0 be the graph on a, b, c, d with edges ab, ac, bc, ad, bd. The graph $B \cup H_0$ is a three-connected, K_5-minor-free minor of G by Lemma 5.4. It is nonplanar since it contains $B + ac + bd$, and it contains the triangle abc. Halin's theorem therefore gives a three-cut X such that $(B \cup H_0) - X$ has at least three components.

Lemma 5.10. *The cut X has, up to interchanging a, b or c, d, one of the forms $\{a, b, w\}$, $\{a, u, v\}$, $\{a, c, w\}$, where $u, v, w \in \mathrm{Int}(B)$. In the latter two cases, exactly two components of $(B \cup H_0) - X$ contain none of a, b, c, d. Both are single vertices p', q' with neighborhood X in G. In each test, p', q' then give two bridges on X adjacent to all of X, each with planar triangle completion, and the test is nonplanar exactly when another bridge is adjacent to all of X, has two attachments α, β and is nonplanar after adding $\alpha\beta$, or has at most one attachment and is nonplanar.*

Proof. Put $A = \{a, b, c, d\}$. If X contains three attachments, the connected interior of B and the remaining attachment make $(B \cup H_0) - X$ connected, a contradiction. If $X \cap A = \{a, b\}$, we have the first form. Otherwise, the added edges keep $A \setminus X$ in one component of $(B \cup H_0) - X$, so at least two components avoid A. These are also components of $B - X$, and Lemma 3.3 makes them exactly two single vertices p', q' with neighborhood X in G. Thus X is a twin triple and, by the assumption of this section, meets both $\{a, b, c\}$ and $\{a, b, d\}$. After possibly interchanging a, b or c, d, this leaves $\{a, u, v\}$, $\{a, c, w\}$, and $\{c, d, w\}$. Suppose $X = \{c, d, w\}$. Then p', q' are leaves of $B[\mathrm{Int}(B)]$, both adjacent to w. Deleting them leaves a connected graph still adjacent to a, b, since neither attachment is adjacent to p' or q'. Contract this graph to w. The five sets

$$\{a, q, r\}, \quad \{b, p\}, \quad \{c, q'\}, \quad \{d, p', z\}, \quad \{w\}$$

are connected and pairwise adjacent, giving a K_5 minor, a contradiction. The last claim follows from Lemma 2.5. □

By Lemma 5.3, the four test graphs are the relevant completions of B. By Lemma 3.7, a reduction of B to an allowed minor therefore leaves B unchanged if it keeps all four tests nonplanar. Several reductions below retain a $K_{3,3}$ subdivision in a component of $B - X$, using the next lemma.

Lemma 5.11. *Let D be a component of $B - X$, and let M be the set of attachments of B in D. Let H_1 be a $K_{3,3}$ subdivision formed by edges of $B[V(D) \cup X]$ and added edges between vertices of $M \cup X$, possibly after identifying a with b. Let $\mathcal{R}$ list required paths between vertices of M or required neighbors at vertices of $M \cup X$, each supplied by one component of $D - M$ and its attaching edges. Then deletions and contractions within $D - M$ give a minor of B with the following properties.*

(i) *It retains H_1, shortened as in the proof of Lemma 3.17 with the branch paths divided at their corners and at the vertices of $M \cup X$, except that a segment that is an edge of B is left unchanged.*

(ii) *It satisfies $\mathcal{R}$, and at most $|\mathcal{R}|$ vertices of $D - M$ lie outside the shortened subdivision.*

(iii) *Each remaining component of $D - M$ keeps its neighbors in $M \cup X$.*

Proof. Divide the branch paths of H_1 at their corners and at the vertices of $M \cup X$, and replace each segment whose ends are joined by an added edge with that edge. We still call the subdivision H_1. Keep every component of $D - M$ met by H_1 and one further component for each requirement not already met. Delete the others. In each kept component C meeting H_1, every component of $C - V(H_1)$ has a neighbor in $C \cap V(H_1)$, since C is connected. Contract each into such a neighbor. Contract each kept component missing H_1 to one vertex. At most $|\mathcal{R}|$ such components were kept, giving the bound in (ii).

Leave any remaining segment that is an edge of B unchanged. Contract the internal vertices of each other segment into an end outside $M \cup X$, or into one vertex if both ends lie in $M \cup X$. If a, b were identified, make these contractions in B with a, b distinct.

Each contraction merges two adjacent vertices of one component of $D - M$. It changes no neighborhood of such a component, keeps every subgraph containing both vertices connected, and keeps the vertices of $M \cup X$ distinct and in the same order on each branch path. Thus every required path and neighbor is preserved, giving (ii) and (iii). If a, b were identified, identifying them again after the reductions gives a graph containing the shortened subdivision. Otherwise the reduced graph itself contains it, together with the added edges. This gives (i). □

After applying Lemma 5.11, delete the edges of D outside the subdivision, except a minimal set preserving the required neighborhoods and connectivity. Each proof below shows that these neighborhoods, this connectivity and the nonplanarity of the retained completion keep all four tests nonplanar. Minimality therefore leaves B unchanged, and Construction 3.18 lists every possible form, with the other fixed vertices marked separately as in that construction.

Lemma 5.12. *Let D be a component of $B - X$ containing exactly one of a, b, c, d, say g.*

(i) *If $N_B(D) = \{s\}$, then $D = \{g\}$ and $gs \in E(B)$.*

(ii) *If $N_B(D) = \{s, t\}$, then each component of $D - g$ is a component of $G - \{g, s, t\}$ with neighborhood $\{g, s, t\}$. There are at most two, and if there are two, both are single vertices.*

Proof. The vertices of $D - g$ are interior vertices of B, so each component of $D - g$ has all its neighbors outside it in $\{g\} \cup N_B(D)$. In (i), three-connectivity leaves no such component, so $D = \{g\}$ and $gs \in E(B)$. In (ii), each component of $D - g$ is a component of $B - \{g, s, t\}$ disjoint from $\{a, b, c, d\}$, and Lemma 3.3 applies. □

5.4.1 A cut of the form $\{a, b, w\}$

This subsection shows that a cut of this form gives $\mathfrak{T}_{56}, \mathfrak{T}_{72}, \mathfrak{T}_{73}$. We first determine the possible forms of the components of $B - X$ and then combine them in Proposition 5.16.

Let $X = \{a, b, w\}$. Every edge of H_0 meets a or b, so $(B \cup H_0) - X = B - X$ has at least three components. At most two contain c or d, so at least one contains neither. By Lemma 3.3, there are at most two such components. Each is a component of $G - X$ with neighborhood X, and if there are two, both are single vertices. Since the interior of B is connected, every component of $\mathrm{Int}(B) - w$ has a neighbor at w. Every component of $B - X$ also has a neighbor at w, since c, d each have an interior neighbor.

Lemma 5.13. *Suppose exactly one component D_0 of $B - X$ contains neither c nor d, and let B_0 be its bridge on X. Then c, d lie in separate components D_c, D_d of $B - X$, and B_0 has one interior vertex or is one of*

$$T(a),\ T(b),\ T(w),\ S(a; b),\ S(a; w),\ S(b; a),\ S(b; w),\ Q(a; b, w),\ Q(b; a, w),\ Q(w; a, b).$$

Proof. Since $B - X$ has at least three components and only D_0 contains neither c nor d, these vertices lie in separate components D_c, D_d. The edge cd is absent by Lemma 5.2, so the components of $G - X$ are D_0, $D_c \cup \{p, q\}$ and $D_d \cup \{r, z\}$. Lemma 3.1 therefore makes the triangle completion of B_0 a three-connected minor of G.

In each of the four tests, B_0 is a bridge on X adjacent to all of X. By Lemma 2.5, the conditions on B_0 depend only on which graphs $B_0 + E$, with $E \subseteq \{ab, aw, bw\}$, are planar. Replacing B_0 by an allowed minor gives an allowed minor of B, since every component of $D_c - c$ and $D_d - d$ remains adjacent to w. These graphs are therefore the relevant completions of B_0, and by Lemma 3.7 every proper allowed minor of B_0 makes at least one previously nonplanar completion planar.

At least one of $B_0 + ab + aw$, $B_0 + ab + bw$ and $B_0 + aw + bw$ is planar. Otherwise, contract a c–w path through D_c and a d–w path through D_d into w. Delete the remaining vertices of D_c, D_d and any edges aw, bw. What remains is B_0 together with p, q, r, z, each now having neighborhood $\{a, b, w\}$. This is a proper minor of G, since c, d were contracted, so it is toroidal. Lemma 4.18 then makes at least one of the three completions planar, a contradiction.

Now suppose $B_0 + ab$ is nonplanar. Then $B_0 + ab + aw$ and $B_0 + ab + bw$ are nonplanar, so $B_0 + aw + bw$ is planar. Apply Proposition 5.7 with $x = w$. The single-vertex form, $T(a), T(b)$ and $Q(w; a, b)$ remain planar after adding ab, so $B_0 = S(w; a)$ or $S(w; b)$. Suppose $B_0 = S(w; a)$, with interior vertices p', q', u, f, where p', q' have neighborhood $\{w, a, u\}$ and $N(f) = \{w, u, b\}$. Contract fb and retain the resulting edge bw. This replaces B_0 by $T(b)$ together with bw. By Lemma 3.10, $T(b) + bw + E$ is planar exactly when $ab \notin E$, which is also the condition for $B_0 + E$ to be planar. Treating bw as an edge bridge on X, Lemma 2.5 therefore keeps all four tests nonplanar. The interior of B remains connected through w, p', q', u, and every other interior component remains adjacent to w.

The attachments a, b retain the interior neighbors p', u, respectively, and the neighbors of c, d are unchanged. Since w is an interior vertex of B, retaining bw creates no edge between its attachments a, b, c, d. Thus this is a proper allowed minor of B keeping all four tests nonplanar, contrary to minimality. Interchanging a, b excludes $S(w; b)$.

Hence $B_0 + ab$ is planar, and at least one completion obtained by adding two edges is planar. Apply Proposition 5.7 with x the attachment incident with both added edges. The forms $S(w; a), S(w; b)$ are excluded because adding ab makes them nonplanar. Taking the three possible choices of x leaves the eleven forms in the statement. □

The next two lemmas give the possible forms of the components of $B - X$ containing c or d.

Lemma 5.14. *Suppose c, d lie in different components D_c, D_d of $B - X$, and put*

$$H_c = B[V(D_c) \cup X] \cup K_3[X].$$

If $H_c + ac + bc$ is planar, then $D_c = \{c\}$ with the edge cw, or $D_c - c$ has at most two vertices, each adjacent to c, w. Otherwise Construction 3.18 gives every possible form of D_c, with one of the following choices.

(i) *Attachments a, b, c, w and added edges ab, aw, bw, ac, bc, with both ac, bc required in the subdivision.*

(ii) *Attachments a, b, c, w, added edges ab, aw, bw and one of ac, bc, which is required in the subdivision.*

(iii) *Attachments x, c, w and added edges xw, xc, with xc required, where x is obtained by identifying a, b. Then x is split back into a, b, assigning each original neighbor of x to a, to b, or to both.*

In all three choices, a segment with ends c, w may be the original edge cw or a path with one interior vertex. The same holds for D_d with d in place of c.

Proof. The component D_c has a neighbor at w. In the tests $B + ac + bd$ and $B + ab + ac + ad$ of Lemma 5.3, the edge ac also joins a to D_c. Thus the bridge containing D_c attaches at a, w, and it attaches at b exactly when b already has a neighbor in D_c in B. In the tests $B + ad + bc$ and $B + ab + bc + bd$, the edge bc gives the same conclusion with a, b interchanged.

Consider the two tests adding ac. If the bridge attaches at all of X, its triangle completion is $H_c + ac$. If it attaches only at a, w, complete it by adding aw. The only other edges in $H_c + ac$ are then those of the path a–b–w, which can be drawn alongside aw in any plane drawing of the completed bridge. Thus in either test the completed bridge is planar exactly when $H_c + ac$ is planar. The same argument gives $H_c + bc$ for the two tests adding bc. At least one component of $B - X$ avoids c, d, and its bridge meets all of X in every test. By Lemma 2.5, a reduction of D_c therefore keeps all four tests nonplanar provided it keeps D_c connected, preserves $N_B(D_c)$ and preserves the planarity or nonplanarity of $H_c + ac$ and $H_c + bc$.

In a plane drawing of B, retain $B[V(D_c) \cup X]$ and a bridge on X avoiding c, d and adjacent to all of X. Contract the interior of that bridge to a vertex and replace the three edges at it by a triangle on X. This gives a plane drawing of H_c. In $B \cup H_0$, contracting the same bridge interior into w supplies aw, bw, and ab, ac, bc are already present. Thus $H_c + E$ is a K_5-minor-free minor of $B \cup H_0$ for every $E \subseteq \{ac, bc\}$. Each component of $D_c - c$ has a neighbor at c since D_c is connected. It is also a component of $\mathrm{Int}(B) - w$, so it has a neighbor at w.

Suppose first that $H_c + ac + bc$ is planar. Then $H_c + ac$ and $H_c + bc$ are planar as well. If neither a nor b has a neighbor in D_c, then $N_B(D_c) = \{w\}$ and Lemma 5.12(i) gives $D_c = \{c\}$ with the edge cw. Otherwise, for each of a, b having a neighbor in D_c, keep one component of $D_c - c$ containing such a neighbor. The neighbor lies outside c since ac, bc are absent from B. Delete the other components of $D_c - c$ and contract each chosen component to one vertex. This leaves at most two vertices besides c, each adjacent to c, w, and preserves $N_B(D_c)$. All four graphs $H_c + E$ remain planar under these operations.

Now suppose $H_c + ac + bc$ is nonplanar. We choose a subdivision according to which of $H_c + ac$ and $H_c + bc$ are nonplanar. Each nonplanar graph under consideration has a $K_{3,3}$ subdivision by Kuratowski's theorem, since it has no K_5 minor.

If both single-edge completions are planar, choose a subdivision in $H_c + ac + bc$. It must use both ac and bc, since deleting either edge leaves a planar graph. If exactly one single-edge completion is nonplanar, interchange a, b if necessary and choose a subdivision in $H_c + ac$. It must use ac, since H_c is planar. Retaining this subdivision also keeps $H_c + ac + bc$ nonplanar.

If both single-edge completions are nonplanar, contracting ab and identifying parallel edges gives the same graph

$$J = (H_c + ac)/ab = (H_c + bc)/ab = (H_c + ac + bc)/ab.$$

Since H_c is connected and planar and contains ab, Lemma 2.7 makes J nonplanar. Write x for the vertex obtained by identifying a, b. Choose a $K_{3,3}$ subdivision in J. It uses xc, since deleting xc leaves the planar graph H_c/ab. Make the reductions in B with a, b distinct, retaining the original edges needed for this subdivision after identifying

a, b. Both reduced single-edge completions then have a nonplanar minor containing the shortened subdivision, so both remain nonplanar.

In each of these three cases, apply Lemma 5.11 to D_c and the chosen subdivision, requiring a neighbor at each of a, b that has one in D_c. Each remaining component of $D_c - c$ keeps its neighbors at c, w. Thus D_c stays connected, its neighborhood in X is preserved, and the interior of B stays connected through w. The shortened subdivision keeps every nonplanar completion nonplanar, and each planar completion remains planar under the reduction. Thus both here and in the all-planar case, all four tests for B remain nonplanar, so minimality leaves B unchanged. Hence D_c already has one of the reduced forms. The three subdivision choices give (i), (ii) and (iii), respectively. The same argument gives the forms for D_d with d in place of c. □

Lemma 5.15. *Suppose c, d lie in the same component D_1 of $B - X$. Then, after interchanging a, b if necessary, $N_B(D_1) = \{w\}$ or $\{a, w\}$, and Construction 3.18 gives every possible form of D_1 with attachments a, c, d, w and added edges aw, ac, ad, requiring the subdivision to use at least one of ac, ad. For a segment with ends c, w or d, w, either the original edge or a path with one interior vertex is allowed.*

Proof. Since $B - X$ has at least three components, Lemma 3.3 gives exactly two other components, each a single vertex adjacent to all of X. If $N_B(D_1) = X$, contracting D_1 to a vertex would give a $K_{3,3}$ minor in the planar graph B. Since D_1 has a neighbor at w, we may interchange a, b and assume $N_B(D_1) = \{w\}$ or $\{a, w\}$.

In $B + ac + bd$, the two added edges join both a and b to D_1. Contracting D_1 to a vertex then gives, with the two single vertices, a $K_{3,3}$ minor. The same holds in $B + ad + bc$. For the other two tests, let B_1 be the bridge of D_1 on X in B, and put

$$H_1 = (B_1 \cup \{a, w\}) + aw.$$

In $B + ab + ac + ad$, both ac, ad join a to D_1, and ab is a separate edge bridge on X. The bridge containing D_1 has exactly two attachments a, w, and adding aw to it gives $H_1 + ac + ad$. The two single-vertex bridges each have triangle completion K_4, so Lemma 2.5(ii) makes this test nonplanar exactly when $H_1 + ac + ad$ is nonplanar. In $B + ab + bc + bd$, if $N_B(D_1) = \{w\}$, the same argument gives the same completed graph with b in place of a. If $N_B(D_1) = \{a, w\}$, adding bc makes the bridge adjacent to all of X, again giving a $K_{3,3}$ minor. Thus it suffices to keep D_1 connected, preserve $N_B(D_1)$ and keep $H_1 + ac + ad$ nonplanar.

An a–w path through one of the single vertices supplies aw outside B_1. Contracting one edge of this path gives H_1 as a minor of the planar graph B. The same contraction in $B \cup H_0$, retaining ac, ad, gives $H_1 + ac + ad$, so it has no K_5 minor. Choose a $K_{3,3}$ subdivision in $H_1 + ac + ad$. It uses at least one of ac, ad, since deleting both leaves H_1.

Each component of $D_1 - \{c, d\}$ has a neighbor at w and at least one of c, d. Since D_1 is connected and cd is absent, the interior of a c–d path in D_1 lies in one such component. Apply Lemma 5.11 to D_1 and this subdivision, requiring a c–d path and, if $a \in N_B(D_1)$, a neighbor at a. This needs at most two further components, keeps D_1 connected and preserves $N_B(D_1)$. The interior of B remains connected through w, and the retained subdivision keeps $H_1 + ac + ad$ nonplanar. As before, all four tests remain nonplanar, so minimality leaves B unchanged, and Construction 3.18 applies as stated. □

Proposition 5.16. *If $X = \{a, b, w\}$, then G is isomorphic to one of $\mathfrak{T}_{56}, \mathfrak{T}_{72}, \mathfrak{T}_{73}$.*

Proof. We distinguish whether c, d lie in different components of $B - X$ or in the same component. Lemmas 5.14 and 5.15 give the possible forms of D_c, D_d and D_1. For each form, discard it if a proper reduction preserves its neighborhood in X, the required connectivity and the planarity conditions established for that case. The check in [Supp. E10] leaves 33 forms for D_c, the same forms for D_d after relabeling, and 54 forms for D_1.

If c, d lie apart, take one retained form for D_c and one for D_d. If exactly one component avoids c, d, include one of the eleven bridges of Lemma 5.13 and a permitted subset of $\{aw, bw\}$. If two components avoid c, d, include the two single vertices adjacent to all of X. In this case aw, bw are absent by Lemma 3.4(a). If c, d lie together, take one retained form for D_1 and include those same two single vertices. Keeping the combinations for which B is planar leaves 1,471 candidates. Of these, the obstruction check in [Supp. E10] leaves only $\mathfrak{T}_{56}, \mathfrak{T}_{72}, \mathfrak{T}_{73}$, each with exactly one component of $B - X$ avoiding c, d. □

5.4.2 A cut of the form $\{a, u, v\}$

Let $X = \{a, u, v\}$, and let p', q' be as in Lemma 5.10, with $N(p') = N(q') = X$. This set is independent by Lemma 3.4(a). Every other component of $B - X$ contains at least one of b, c, d and has a neighbor at u or v, since these attachments have interior neighbors and the interior of B is connected. None has neighborhood X, since together with p', q' it would give three bridges adjacent to all of X, contrary to the planarity of B.

We distinguish the cases according to whether b, c, d lie in one, two or three components of $B - X$. The only edges of H_0 that survive deletion of X are bc and bd, so

$$(B \cup H_0) - X = (B - X) + bc + bd.$$

In $(B \cup H_0) - X$ the three vertices therefore lie in one component (Figure 14).



Figure 14: The ovals show the components of $B - X$ containing b, c, d when the three lie apart. Adding bc and bd joins these components, while p' and q' remain isolated.

Lemma 5.17. *If one component D of $B - X$ contains b, c, d, then G is isomorphic to one of $\mathfrak{T}_{40}, \mathfrak{T}_{53}, \mathfrak{T}_{56}, \mathfrak{T}_{73}$.*

Proof. If u had no neighbor in D, its only neighbors in G would be p', q', contrary to three-connectivity. The same holds for v. If a also had a neighbor in D, contracting D to a vertex would give a $K_{3,3}$ minor in the planar graph B, with p', q' and the contracted vertex adjacent to all of X. Thus $N_B(D) = \{u, v\}$. Each of the four tests of Lemma 5.3 adds an edge from a to D. The same contraction therefore makes each test nonplanar as long as D stays connected and keeps neighbors at both u and v.

Let $C_1, \ldots, C_m$ be the components of $D - \{b, c, d\}$ and contract each C_i to one vertex x_i. Keep a spanning tree of the resulting D. Each C_i had a neighbor at u or v, since the interior of B is connected and these are the only vertices through which C_i can reach the rest of that interior. Retain an inclusion-minimal set of edges from D to $\{u, v\}$ such that each x_i still has a neighbor at u or v and both u and v still have a neighbor in D. The tree gives each of b, c, d an interior neighbor, and every x_i is joined to the rest of the interior through u or v. This gives an allowed minor of B keeping all four tests nonplanar. By minimality each C_i was already a single vertex, D was already this tree, and its edges to u, v were already minimal.

The tree D is bipartite with parts $\{b, c, d\}$ and $\{x_1, \ldots, x_m\}$. Suppose x_i is a leaf. Its only possible neighbors outside D are u, v, so minimum degree three in G gives both ux_i and vx_i. If $m = 1$, the only way to join b, c, d in the tree is to join all three to x_i, contrary to its being a leaf. Thus there is another x_j, adjacent to one of u, v, say u. Deleting ux_i leaves x_i adjacent to v and u adjacent to x_j, contrary to the minimal choice of edges. Hence every x_i has degree at least two in D. Each edge of the tree has exactly one end among the x_i, so

$$m + 2 = |E(D)| = \sum_{i=1}^{m} \deg_D(x_i) \geq 2m.$$

Thus m is one or two.

If $m = 1$, then D is the star with center x_1 and leaves b, c, d. The center is adjacent to both u, v, or to one of them while one of b, c, d is adjacent to the other. If $m = 2$, the degrees of x_1, x_2 in D sum to four, so both have degree two. Their neighborhoods in $\{b, c, d\}$ meet in one vertex and together contain all three. Thus D is the path $g_1x_1g_2x_2g_3$ for some ordering g_1, g_2, g_3 of b, c, d. The vertices x_1, x_2 are adjacent to different vertices of $\{u, v\}$, or both are adjacent to one and one of b, c, d is adjacent to the other. These choices give the 31 forms checked in [Supp. E11], leaving exactly $\mathfrak{T}_{40}, \mathfrak{T}_{53}, \mathfrak{T}_{56}, \mathfrak{T}_{73}$. □

Lemma 5.18. *Suppose b, c, d do not all lie in one component of $B - X$, and let $D_g \neq \{g\}$ be a component containing exactly one of them, g. Then $N_B(D_g) = \{s, t\}$ for some $s, t \in X$. Either $D_g - g$ consists of two single vertices with neighborhood $\{g, s, t\}$, or it is connected and its bridge B_g on $\{g, s, t\}$ is one of nine forms, namely one interior vertex, $T(g), T(s), T(t)$, $S(s; t), S(t; s)$, $Q(g; s, t)$ and the two four-cycle forms of Lemma 3.14.*

Proof. First take $g = b$, so D_b contains b and neither c nor d. We use c as an attachment in another component of $B - X$. When $g = c$ or d, use g in place of b throughout the local argument and b for the attachment in the other component. Since $D_b \neq \{b\}$, Lemma 5.12(i) excludes a single neighbor in X. It cannot have all three, since contracting D_b would give a $K_{3,3}$ minor in B together with p', q' and X. Thus $N_B(D_b) = \{s, t\}$ for two vertices $s, t \in X$. By Lemma 5.12(ii), each component of $D_b - b$ has neighborhood $\{b, s, t\}$ in G. There are at most two, and if there are two, both are single vertices. This gives the first outcome.

It remains to consider when $D_b - b$ is connected. Let B_b be its bridge on $\{b, s, t\}$. Any original edges bs, bt lie outside B_b. Consider the four tests of Lemma 5.3,

$$B + ac + bd, \qquad B + ad + bc, \qquad B + ab + ac + ad, \qquad B + ab + bc + bd.$$

In each test the path $sp't$ joins s to t outside B_b. If B_b is the only bridge adjacent to all of b, s, t, the bridge containing this path has attachments s, t, so Lemma 2.5(iii) includes st in the completion of B_b. A second bridge adjacent to all three requires the full triangle completion by part (ii), and three such bridges already make the test nonplanar by part (i). The four possible completions are $B_b + st + E$ with $E \subseteq \{bs, bt\}$. Retaining B_b and the path $sp't$ in B and contracting sp' gives $B_b + st$, so this completion is planar.

The component of $B - X$ containing c has a neighbor at u or v. Interchange u, v if necessary and choose a path P from u to c with all its vertices after u in that component. The edges ba, cb of H_0, together with P and the edges from p', q' to X, give the cycle

$$b - a - p' - v - q' - u \xrightarrow{P} c - b.$$

The cycle contains b, s, t since it passes through b and all of X. It avoids the interior of B_b since P runs through the component containing c. Retain B_b and this cycle, and shorten each of the three arcs between b, s, t to one edge. This gives the triangle completion of B_b as a minor of $B \cup H_0$, hence of G. Since $D_b - b$ is a component of $G - \{b, s, t\}$, Lemma 3.1 also makes this completion three-connected.

If the triangle completion is planar, Lemma 3.2 makes $D_b - b$ a single vertex. Suppose it is nonplanar. An allowed reduction of B_b keeps its interior connected and adjacent to b, s, t. At least one of s, t lies in $\{u, v\}$, so the reduced interior stays joined to the rest of the interior of B. Thus an allowed minor of B_b gives an allowed minor of B, and the four completions $B_b + st + E$ are the relevant completions of B_b. By Lemma 3.7, Proposition 3.15 applies with s, t, b in the roles of x, y, z, and gives the other eight forms. □

Lemma 5.19. *Suppose b, c, d lie in two components D_1, D_2 of $B - X$, where D_1 contains x, y and D_2 contains g, with bridges B_1, B_2 on X. Put $H_1 = B_1 + \{a, u\} + au$. Up to interchanging u, v and c, d, the neighborhoods are as in one of the following rows. All four tests are nonplanar exactly when the condition in that row holds.*

	D_1 contains	$N_B(D_1)$	$N_B(D_2)$	condition
(1)	b, c	$\{u\}$ or $\{a, u\}$	$\{u, v\}$	always
(2)	b, c	$\{u\}$ or $\{a, u\}$	$\{v\}$ or $\{a, v\}$	$B_2 + av + ad$ nonplanar, or both $H_1 + bc$ and $H_1 + ab + ac$ nonplanar
(3)	b, c	$\{u, v\}$	$\{u, v\}$	always
(4)	b, c	$\{u, v\}$	$\{v\}$ or $\{a, v\}$	$B_1 + uv + bc$ or $B_2 + av + ad$ nonplanar
(5)	c, d	at least one is $\{u, v\}$		always
(6)	c, d	$\{u\}$ or $\{a, u\}$	$\{v\}$ or $\{a, v\}$	$B_1 + au + ac + ad$ or $B_2 + av + ab$ nonplanar

Proof. Since p', q' are the only neighbors of u, v outside $D_1 \cup D_2$, minimum degree three gives each of u, v a neighbor in D_1 or D_2. Each test of Lemma 5.3 adds an edge from a to D_1 or D_2. Thus any test joining D_1, D_2 gives a third bridge adjacent to all of X and is nonplanar.

Suppose first that D_1 contains b, c. The tests containing bd join D_1, D_2. In the two remaining tests, $B + ad + bc$ and $B + ab + ac + ad$, the edge ad joins a to D_2. If $N_B(D_2) = \{u, v\}$, its bridge attaches at all of X in both tests, giving the first and third rows. Otherwise, up to interchanging u, v, its attachments in both tests are a, v, so its completion is $B_2 + av + ad$ by Lemma 2.5(ii).

If $N_B(D_1) = \{u, v\}$, adding ab, ac makes its bridge adjacent to all of X, while adding bc leaves it with attachments u, v and completion $B_1 + uv + bc$. The two tests are therefore nonplanar exactly when $B_1 + uv + bc$ or $B_2 + av + ad$ is nonplanar, giving the fourth row.

Otherwise D_1 meets only u among u, v. In $B + ad + bc$, if D_1 meets a, adding au completes its bridge to $H_1 + bc$. If it meets only u, then $H_1 + bc$ differs from $B_1 + bc$ only by the leaf a at u. In $B + ab + ac + ad$, its bridge has attachments a, u and completion $H_1 + ab + ac$. Thus both tests are nonplanar exactly when $B_2 + av + ad$ is nonplanar or both $H_1 + bc$ and $H_1 + ab + ac$ are nonplanar, giving the second row.

Now suppose D_1 contains c, d. Every test containing bc or bd joins D_1, D_2. In the remaining test $B + ab + ac + ad$, the edges ac, ad join a to D_1 and ab joins a to D_2. If either component already meets both u, v, its bridge now attaches at all of X, giving the fifth row. Otherwise, up to interchanging u, v, the bridges have attachments a, u and a, v, respectively. Completing them by au and av gives $B_1 + au + ac + ad$ and $B_2 + av + ab$. The test is nonplanar exactly when at least one of these is nonplanar, giving the sixth row. □

We next restrict D_2.

Lemma 5.20. *Suppose b, c, d lie in two components D_1, D_2 of $B - X$, as in Lemma 5.19. If $D_2 = \{g\}$, then g is adjacent in B to u, to v, or to both. Otherwise $D_2 - g$ is connected and its bridge has one of the nine forms of Lemma 5.18.*

Proof. Suppose for the moment that $d \in D_2$. Either $D_2 = \{d\}$, or Lemma 5.18 gives two neighbors $s, t \in X$ and describes $D_2 - d$. If $D_2 - d$ consists of two single vertices, they both have neighborhood $\{d, s, t\}$ in B. Delete one of them from B. The other keeps an interior neighbor for d, preserves the neighbors of D_2 in X, and can replace the deleted vertex on any path through the interior of B. Thus the deletion is allowed. Every test joining D_1, D_2 or making D_2 adjacent to all of X still gives a third bridge adjacent to all of X. In every remaining test the completion of the bridge containing D_2 is a subgraph of K_5 minus the edge between the two single vertices, so it was planar before the deletion. The other bridges are unchanged, and all four tests remain nonplanar, contrary to minimality. The same argument applies with b in place of d. Thus two possibilities remain. If D_2 consists of its lone vertex from $\{b, c, d\}$, that vertex is adjacent in B to u, to v, or to both. Otherwise deleting that vertex from D_2 leaves a connected interior whose bridge has one of the nine forms of Lemma 5.18. □

In rows two and four of Lemma 5.19, the completion listed for B_2 is $B_2 + av + ad$. In row six it is $B_2 + av + ab$. The next three lemmas treat these rows when the listed completion of B_2 is planar. We treat rows four and six first, since each requires one nonplanar completion of B_1. Row two requires two.

Lemma 5.21. *In row four of Lemma 5.19, suppose $B_2 + av + ad$ is planar. Then Construction 3.18 lists every possible B_1, with attachments b, c, u, v, added edges bc, uv, required edge bc, and permitted original edges bu, bv, cu, cv, subject to D_1 and $(B_1 - \{b, c\}) + uv$ being connected and the bound of two further components.*

Proof. In row four, $N_B(D_1) = \{u, v\}$. The path $up'v$ supplies uv outside B_1, so $B_1 + uv$ is a planar minor of B. Retaining bc as well in $B \cup H_0$ gives $B_1 + uv + bc$ as a K_5-minor-free minor. Row four requires this graph to be nonplanar. Retain a $K_{3,3}$ subdivision in it. The subdivision must use bc, since $B_1 + uv$ is planar. Each component of $D_1 - \{b, c\}$ has a neighbor at b or c by connectedness of D_1, and at u or v by connectedness of the interior of B. The subdivision meets at least one such component, since it has six corners. The interior of a b–c path in D_1 lies in a single such component, since bc is absent from B. Apply Lemma 5.11, requiring this path and, if the components met by the subdivision meet only one of u, v, a neighbor at the other. This needs at most two further components. The reduction keeps $N_B(D_1) = \{u, v\}$ and keeps both D_1 and $(B_1 - \{b, c\}) + uv$ connected. The retained subdivision keeps all four tests nonplanar by row four, so minimality leaves B unchanged, and Construction 3.18 applies as stated. □

Lemma 5.22. *In row six of Lemma 5.19, suppose $B_2 + av + ab$ is planar. Then Construction 3.18 lists every possible B_1, with attachments a, c, d, u, added edges au, ac, ad, and permitted original edges cu, du, subject to D_1 and $B_1 - \{a, c, d\}$ being connected and the bound of one further component.*

Proof. In row six, D_1 contains c, d and meets only u among u, v. The path $ap'u$ supplies au outside B_1, so $B_1 + au$ is a planar minor of B and $B_1 + au + ac + ad$ is a K_5-minor-free minor of $B \cup H_0$. Row six requires the latter graph to be nonplanar. Retain a $K_{3,3}$ subdivision in it. It uses ac or ad. If it uses no original edge from a to D_1, delete all such edges. The vertex a keeps its interior neighbor p'. Each test containing bc or bd still joins D_1, D_2 into a bridge adjacent to all of X, and the subdivision persists as a minor of $B + ab + ac + ad$ using $ap'u$. By minimality, this deletion removes no edge. Thus if D_1 meets a, the subdivision already preserves that neighbor. Every component of $D_1 - \{c, d\}$ has a neighbor at u, and at most one further component is needed to retain a c–d path. By Lemma 5.11 this keeps D_1 and $B_1 - \{a, c, d\}$ connected and preserves $N_B(D_1)$. All four tests remain nonplanar, so minimality leaves B unchanged, and Construction 3.18 applies as stated. □

Lemma 5.23. *In row two of Lemma 5.19, suppose $B_2 + av + ad$ is planar. Then $D_2 = \{d\}$ with the edge dv, and Construction 3.18 lists every possible B_1, with attachments a, b, c, u, added edges bc, au, required edge bc, and permitted original edges bu, cu.*

Proof. In row two, put $H_1 = B_1 + \{a, u\} + au$ as in Lemma 5.19. Both $H_1 + bc$ and $H_1 + ab + ac$ are nonplanar. Reduce B_2 to the edge dv. This edge is present if $D_2 = \{d\}$. Otherwise $D_2 - d$ is connected by Lemma 5.20 and has neighbors at d and v. Contract it into v and delete any resulting edge av. The interior stays connected and d keeps its interior neighbor v. In both tests containing bc, the path $ap'u$ supplies au, giving $H_1 + bc$ as a minor. It likewise gives $H_1 + ab + ac$ as a minor of $B + ab + ac + ad$. In $B + ac + bd$, the paths $ap'u$ and $aq'vdb$ supply au and ab, respectively, giving $H_1 + ab + ac$ again. Thus all four tests remain nonplanar, so $D_2 = \{d\}$ with the edge dv by minimality.

Using $ap'u$ for au gives H_1 as a planar minor of B and $H_2 = H_1 + ab + ac + bc$ as a K_5-minor-free minor of $B \cup H_0$. Each component of $D_1 - \{b, c\}$ has a neighbor at u, and each of b, c has an interior neighbor in one of

these components or at u. Thus $H_1 - \{a, b, c\}$ is connected and has neighbors at a, b, c, using au at a. In particular, $H_2 - a$, $H_2 - \{a, b\}$ and $H_2 - \{a, c\}$ are connected. If at most two vertices other than a are deleted, the surviving vertices of $\{a, b, c, u\}$ stay connected through a. A component disjoint from these vertices would also be separated in G by the same deletion, contrary to three-connectivity. Hence H_2 is two-connected and every two-cut contains a and avoids b, c. Applying Lemma 3.16 with b, c, a in the roles of x, y, z gives a $K_{3,3}$ subdivision H_3 in $H_1 + bc$ using bc for which $H_3 - bc + ab + ac$ is nonplanar.

Shorten the branch paths of H_3 as in Lemma 3.17, dividing them at their corners and at a, b, c, u wherever these vertices occur. For a segment from u to b or c, retain the original edge if it is an edge and one interior vertex otherwise. Each component of $D_1 - \{b, c\}$ meets u, so contract its remaining vertices onto retained vertices or onto u. Keep a minimal forest joining the retained interior vertices to u, the edges from p', q' to X, and dv. The interior stays connected and each attachment keeps an interior neighbor. Any edge au created by contraction can be deleted from B, since $ap'u$ still supplies it in H_1. The shortened subdivision keeps $H_1 + bc$ nonplanar. Shortening also preserves the position or absence of a on its branch paths, so Lemma 2.6 keeps $H_1 + ab + ac$ nonplanar. Thus all four tests remain nonplanar and minimality leaves B unchanged.

In applying Construction 3.18, keep the subdivisions H for which $H - bc + ab + ac$ is nonplanar, allowing every position of u, including outside H, the permitted original edges bu, cu, and every minimal connecting forest described above. At most one segment of H of length two joins a to b or c. Two would form a cycle, using bc if their other ends differ, with corners only among a, b, c. Every cycle in a $K_{3,3}$ subdivision contains at least four corners. □

Proposition 5.24. *If $X = \{a, u, v\}$, then G is isomorphic to one of $\mathfrak{T}_{40}, \mathfrak{T}_{51}, \mathfrak{T}_{53}, \mathfrak{T}_{56}, \mathfrak{T}_{72}, \mathfrak{T}_{73}$.*

Proof. If b, c, d all lie in one component of $B - X$, Lemma 5.17 shows that G is isomorphic to one of $\mathfrak{T}_{40}, \mathfrak{T}_{53}, \mathfrak{T}_{56}, \mathfrak{T}_{73}$. It remains to consider when b, c, d lie in two or three components of $B - X$.

Suppose first that b, c, d lie in distinct components D_b, D_c, D_d of $B - X$. If $D_b = \{b\}$, then b is adjacent to u, to v, or to both, since ab is absent and b has an interior neighbor. Otherwise Lemma 5.18 gives $N_B(D_b) = \{s, t\}$ for two vertices $s, t \in X$. Either $D_b - b$ consists of two single vertices adjacent to b, s, t, or $D_b - b$ is connected and its bridge on $\{b, s, t\}$ has one of the nine forms of that lemma. For each choice of $\{s, t\}$, include each permitted subset of the original edges bs, bt, with ab excluded. Make the same choices for D_c, D_d, with c, d in place of b and ac, ad excluded, respectively.

Together with p', q' and Ψ_2, these choices give every possible G in this case. The DRAT coverage check in [Supp. E12] shows that each combination is a subgraph of a recorded toroidal graph or contains a recorded nontoroidal graph as a subgraph. Since G is nontoroidal, it must contain one of the latter graphs, and by minimality it equals that graph. This leaves $\mathfrak{T}_{51}$ and $\mathfrak{T}_{72}$. In neither does any of $D_b - b, D_c - c, D_d - d$ consist of two single vertices.

Now suppose b, c, d lie in two components of $B - X$. Let D_1 contain two of them and let D_2 contain the third. Let B_1, B_2 be their bridges on X. Up to interchanging c, d, either D_1 contains b, c and D_2 contains d, or D_1 contains c, d and D_2 contains b. These are the first four rows and the last two rows of Lemma 5.19, respectively. Lemma 5.20 gives the possible forms of D_2. When the listed completion of B_2 is planar, Lemmas 5.21, 5.22 and 5.23 give those of B_1 in rows four, six and two.

It remains to consider rows one, three and five, where the neighborhoods alone suffice, and rows two, four and six when the listed completion of B_2 is nonplanar. Delete the edges from a to D_1. In rows one, three and five, each test either joins D_1, D_2 and supplies an edge to a, or joins a directly to a component already meeting both u, v. In rows two, four and six, any test joining D_1, D_2 still supplies an edge to a as well. In the remaining tests of rows two, four and six the nonplanar completion of B_2 suffices.

When D_1 contains b, c, choose a b–c path in D_1 and contract the component of $D_1 - \{b, c\}$ containing its interior to one vertex m. This gives the path bmc, and m has a neighbor at u or v. If D_1 also met the other vertex and m misses it, retain an edge from b or c to that vertex. If neither edge was present, contract a further component supplying the missing neighbor into b or c, whichever it meets, and delete any resulting edge bc. A minimal choice of edges from bmc to $\{u, v\}$ preserving these neighbors gives

$$mu, \quad mv, \quad mu + mv, \quad mu + bv, \quad mu + cv, \quad mv + bu, \quad mv + cu.$$

When D_1 contains c, d, the same reduction gives the path cmd and the same list with b, c replaced by c, d, respectively.

In rows one, three and five, contract D_2 to its lone attachment vertex, keeping its neighbors in $\{u, v\}$ and deleting any new edge to a. Each test still has a third bridge adjacent to all of X. In the other rows under consideration, retain B_2 with its permitted original edges. In both instances the interior stays connected through

u, v, p', q', and all four attachments keep an interior neighbor. All four tests remain nonplanar, so minimality forces B to have one of the forms just obtained.

As in Proposition 5.16, discard any form having a proper allowed minor that preserves its neighborhood, the required connectivity and which completions in its row are planar. For each row, combine the retained forms of B_1, B_2 with all permitted original edges, and restore p', q' at X and Ψ_2 at a, b, c, d. The checks in [Supp. E13] exclude all 1,229 resulting candidates. Thus the two-component case gives no obstruction, and the one-component and three-component cases give precisely the six graphs in the statement. □

5.4.3 A cut of the form $\{a, c, w\}$

This subsection shows that a cut of this form cannot occur. Lemmas 5.25 and 5.26 place b, d in separate components of $B - X$ and give the neighborhoods of these components. Proposition 5.27 then excludes the forms that remain.

Let $X = \{a, c, w\}$, and let p', q' be as in Lemma 5.10, with $N(p') = N(q') = X$. The edges aw, cw are absent by Lemma 3.4(a). Every other component of $B - X$ contains b or d and has a neighbor at w, since these attachments have interior neighbors and the interior of B is connected and meets X only at w. Also, c has a neighbor in one of these components. Otherwise its neighbors are exactly p, q, p', q', contrary to Lemma 5.5 applied to the twin pairs with neighborhoods $\{a, b, c\}$ and $\{a, c, w\}$.

Lemma 5.25. *If b, d lie in separate components D_b, D_d of $B - X$, then $N_B(D_b) = \{c, w\}$ and $N_B(D_d)$ is $\{w\}$ or $\{a, w\}$.*

Proof. The component D_d has no neighbor at c. Otherwise, use p, q and p', q' as the twin pairs, with neighborhoods $\{a, b, c\}$ and $\{a, c, w\}$. The bridge of the new Ψ_2 containing D_d and r, z has all four attachments a, b, c, w. By Lemma 5.2, bw is absent and every other bridge is a single vertex with neighborhood $\{a, b, c\}$ or $\{a, c, w\}$. Thus every component of $D_b - b$ is a single vertex. Since D_b is connected, each is adjacent to b and therefore has neighborhood $\{a, b, c\}$. Together with the absence of bw, this gives no neighbor of D_b at w, a contradiction. Hence c has a neighbor in D_b. The component D_b has none at a, since otherwise D_b, p', q' give three bridges adjacent to all of X in the planar graph B. Both components have a neighbor at w, so the stated neighborhoods follow. □

Lemma 5.26. *The vertices b, d do not lie in one component of $B - X$.*

Proof. Suppose b, d lie in one component D_1, with bridge B_1 on X. It is the only component other than p', q', so it has neighbors at c, w. It has none at a, since otherwise D_1, p', q' make B nonplanar. Thus a is adjacent only to the six twins with neighborhoods $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, w\}$, and deleting a and these six vertices leaves B_1. Now $B_1 - \{c, w\} = D_1$ is connected, and $B_1 - \{b, c, d\}$ is the interior of B with the leaves p', q' removed, so it is connected. The neighbor of c in D_1 is not b or d, since bc, cd are absent. Hence

$$B_1 - \{c, w\} \text{ connected}, \quad B_1 - \{b, c, d\} \text{ connected}, \quad N_{B_1}(c) \setminus \{b, c, d, w\} \neq \varnothing. \tag{2}$$

Only bw, dw are possible edges among b, c, d, w. In $B + ad + bc$, $B + ab + ac + ad$ and $B + ab + bc + bd$, the added edge ad or ab makes the bridge containing D_1 a third bridge adjacent to all of X. In the remaining test $B + ac + bd$, that bridge is $B_1 + bd$, with attachments c, w. This test is nonplanar exactly when $B_1 + bd + cw$ is nonplanar. The latter is a K_5-minor-free minor of G, using brd and $cp'w$, while $B_1 + cw$ is a planar minor of B, using $cp'w$.

Retain a $K_{3,3}$ subdivision of $B_1 + bd + cw$. It uses bd, since deleting this edge leaves the planar graph $B_1 + cw$. The vertices of this subdivision outside $\{b, c, d, w\}$ may lie in different components of $B_1 - \{b, c, d, w\}$. By (2), every such component has a neighbor at w and at b or d. Retain a b–d path in D_1 and a neighbor of c outside $\{b, c, d, w\}$. Since D_1 avoids c, w, the internal vertices of the path lie in one component of $B_1 - \{b, c, d, w\}$. The chosen neighbor of c lies in one such component as well. Thus at most two further components are needed. By Lemma 5.11, contract each to one vertex and keep a minimal set of attaching edges satisfying (2). The retained subdivision keeps $B + ac + bd$ nonplanar. In the other three tests, the bridge containing D_1 is still adjacent to all of X. The conditions in (2) keep the interior of B connected and give every attachment an interior neighbor. Thus the reduction is allowed and keeps all four tests nonplanar, so minimality leaves B unchanged. Construction 3.18 lists every possible B_1, with attachments b, c, d, w, added edges bd, cw, the original edges bw, dw permitted on the subdivision, the requirements (2), and at most two further interior components. Adding a and the three pairs of twins gives the 5,819 candidates in [Supp. E16]. Each has nonplanar B, a K_5 minor, a cut of at most two vertices, or a proper nontoroidal minor. □

Proposition 5.27. *The cut X cannot have the form $\{a, c, w\}$.*

Proof. By Lemmas 5.25 and 5.26, b, d lie in separate components D_b, D_d with the stated neighborhoods. Put $B_b = B[V(D_b) \cup \{c, w\}]$, with attachments b, c, w. Its interior $D_b - b$ is nonempty, since b is not adjacent to c. By Lemma 5.12(ii), this interior is connected or consists of two single vertices with neighborhood $\{b, c, w\}$. In the second case, the triangle completion of B_b is planar.

First suppose $N_B(D_d) = \{w\}$. Lemma 5.12(i) gives $D_d = \{d\}$ with the edge dw. In $B + ab + ac + ad$ and $B + ab + bc + bd$, the edge ab makes the bridge containing D_b a third bridge adjacent to all of X. In $B + ac + bd$ and $B + ad + bc$, that bridge has attachments c, w. These tests are therefore nonplanar exactly when $B_b + cw + bw$ and $B_b + cw + bc$, respectively, are nonplanar, with the path bdw supplying bw in the first. Both completions are nonplanar, so $D_b - b$ is connected. The graph $B_b + cw$ is a planar minor of B, using $cp'w$, so bw is absent. Thus b, c, w are independent, and each has a neighbor in $D_b - b$.

The triangle completion of B_b is three-connected by Lemma 3.1, since $D_b - b$ is a component of $G - \{b, c, w\}$. It is a minor of G, using the internally disjoint paths bpc, $bqap'w$ and $cq'w$ for bc, bw, cw, so it has no K_5 minor. While D_b keeps its neighbors at c, w, the four tests depend on B_b only through $B_b + cw + bw$ and $B_b + cw + bc$. Replacing B_b by a proper allowed minor gives an allowed minor of B, since its interior stays adjacent to w, the vertices a, c keep p', q', and d keeps w. By minimality, no such minor keeps both completions nonplanar. Lemma 3.14, with c, w, b as x, y, z, therefore makes B_b either $Q(b; c, w)$ or one of the two four-cycle forms. The enumeration [Supp. E14] applies Construction 3.18 after identifying c, w to a vertex x, with attachments b, x and added edge bx. It then splits x back into c, w, assigning each original edge at x to one of them and adding an edge to an interior vertex at whichever has no interior neighbor. With Ψ_2, p', q' and dw, the three forms above are among its 24 candidates. Each has a cut of at most two vertices or a proper nontoroidal minor.

Now suppose $N_B(D_d) = \{a, w\}$. Put $B_d = B[V(D_d) \cup \{a, w\}]$, with attachments d, a, w. Its interior $D_d - d$ is nonempty, since d is not adjacent to a. In $B + ac + bd$, $B + ab + ac + ad$ and $B + ab + bc + bd$, the bridge containing D_b is a third bridge adjacent to all of X. In $B + ad + bc$, the bridges containing D_b, D_d are $B_b + bc$ and $B_d + ad$, with attachments c, w and a, w. This test is nonplanar exactly when $B_b + cw + bc$ or $B_d + aw + ad$ is nonplanar. Both are K_5-minor-free minors of G, using $cp'w, bpc$ for the added edges of the first and $aq'w, ard$ for those of the second. The graphs $B_b + cw$ and $B_d + aw$ are planar minors of B, using $cp'w$ and $aq'w$.

Choose one of these nonplanar completions. Its bridge has connected interior, since Lemma 5.12(ii) would otherwise give two single vertices whose triangle completion is planar. Retain a $K_{3,3}$ subdivision of the chosen completion. Since $B_b + cw$ and $B_d + aw$ are planar, the subdivision uses bc or ad, respectively. Reduce the chosen bridge while retaining the subdivision, its connected interior and an interior neighbor at each attachment. In the other bridge, keep one interior component, delete the others and contract the retained component to one vertex. Its neighborhood is the full attachment triple by Lemma 5.12(ii). Both bridges keep their attachment sets, and their interiors remain connected through w, so this gives an allowed minor of B. The chosen completion stays nonplanar, and in the other three tests the bridge containing D_b is still adjacent to all of X. Thus all four tests remain nonplanar, so minimality leaves B unchanged. Construction 3.18 lists every possible B_b with attachments b, c, w, added edges cw, bc and required edge bc, or every possible B_d with attachments d, a, w, added edges aw, ad and required edge ad. An original edge bw or dw may lie on the retained subdivision, so that segment is taken both as an edge and as a path with one interior vertex. The records in [Supp. E15] include both choices of the nonplanar completion. Each of the 156 candidates has nonplanar B, a K_5 minor, a cut of at most two vertices, or a proper nontoroidal minor, a contradiction. □

5.5 Bridges with three attachments: number and planar completions

For the rest of this section, suppose no choice of twin pairs gives a bridge with four attachments. Each component of $G - V(\Psi_2)$ then gives a bridge with three attachments. We distinguish two cases according to whether every such bridge has planar triangle completion. If all the triangle completions are planar, Lemma 5.28 gives the three obstructions $\mathfrak{T}_9, \mathfrak{T}_{10}, \mathfrak{T}_{16}$. If at least one triangle completion is nonplanar, its component has more than one vertex, and the same lemma leaves at most four components outside Ψ_2. We show that each of their bridges has a planar two-edge completion. Section 5.6 uses these completions to determine the possible bridges, and Proposition 5.33 gives $G \cong \mathfrak{T}_{15}$.

In both cases, the triangle completion of each bridge is three-connected by Lemma 3.1. It is also a minor of G. Indeed, up to interchanging a, b or c, d, the attachment set is $\{a, b, c\}$ or $\{a, c, d\}$. For $\{a, b, c\}$, contract p into a and q into b to obtain all three triangle edges. For $\{a, c, d\}$, shorten the paths apc, ard and $cqbzd$ to ac, ad and cd, respectively. These paths have disjoint interiors outside the bridge. In either case, the triangle completion has no K_5 minor.

Lemma 5.28. *If the bridge of every component outside Ψ_2 has planar triangle completion, then G is isomorphic to one of $\mathfrak{T}_9, \mathfrak{T}_{10}, \mathfrak{T}_{16}$. If some bridge has nonplanar triangle completion, then its component has more than one vertex and there are at most four components outside Ψ_2.*

Proof. Put $A = \{a, b, c, d\}$. Each component of $G - A$ has at least three neighbors in A by three-connectivity and at most three by our assumption on the bridges. Four of the components are the single vertices p, q, r, z. The others are precisely the components outside Ψ_2.

If $G - A$ has at least nine components, retain nine, including p, q, r, z, and contract each to one vertex. Delete cd if present. The resulting simple bipartite minor has 13 vertices and 27 edges. By Euler's formula a simple bipartite graph on 13 vertices embedded in the torus has at most $2 \cdot 13 = 26$ edges. The minor is therefore nontoroidal, so minimality forces it to be G. Thus $G - A$ has at most nine components, and equality requires every component to be a single vertex.

Suppose first that every triangle completion is planar. By Lemma 3.2, every component outside Ψ_2 is a single vertex. Thus G is determined by the numbers of vertices with neighborhoods $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and $\{b, c, d\}$, together with the presence or absence of cd. At least two vertices have each of the first two neighborhoods, and the four counts sum to at most nine. These give the 252 choices checked in [Supp. E17]. Every graph except $\mathfrak{T}_9, \mathfrak{T}_{10}, \mathfrak{T}_{16}$ has a cut of at most two vertices, is toroidal, or has a proper nontoroidal minor. In all three remaining graphs, cd is absent.

Now suppose some bridge has nonplanar triangle completion. If its component were a single vertex, that vertex and its three attachments would form K_4 upon adding the triangle edges which is planar and hence the component must have more than one vertex. There are then at most eight components of $G - A$. Removing the four chosen twins from this count leaves at most four components outside Ψ_2. □

It remains to consider the case where at least one bridge has nonplanar triangle completion. For a bridge B with attachments x, y, t, the full triangle completion is $B + xy + xt + yt$. We show that the attachments can always be labeled so that $B + xy + xt$ is planar, even when adding the third edge yt makes the completion nonplanar. We first obtain the two edges from the boundary of a disk face, and then apply this argument to a minor of G.

Lemma 5.29. *Suppose a graph consists of Ψ_2, bridges with three attachments among a, b, c, d, and possibly cd, with cd or a bridge joining c, d. If it has a torus embedding in which a closed regular neighborhood of Ψ_2 has genus one, then each added bridge B has a planar completion $B + xy + xt$ for some labeling of its attachments.*

Proof. Fix such a torus embedding. By Lemma 2.1, every complementary region of the closed regular neighborhood of Ψ_2 is a disk. The edge cd or the bridge joining c, d lies in one such region, so some boundary walk contains both c and d. For every genus-one rotation of Ψ_2 with such a walk, the calculation in [Supp. R4] shows that no two attachment labels alternate on the boundary of any face containing at least three distinct labels.

Choose an added bridge B. Its interior is connected, so it lies in one disk face F, with all three attachments on the boundary walk of F. Place a copy of an attachment at each boundary appearance used by B, and draw a cycle through the copies in their boundary order. Outside the cycle, join the copies of each attachment by a tree. To draw these trees, regard the outside as a second disk and join successive copies of each label by chords, omitting the closing chord. A crossing between chords for two different labels would give four appearances alternating around the boundary of F. Thus the trees are disjoint.

Contract each tree to a single vertex. This restores B and turns the cycle into a connected graph on the three attachments. Keep a spanning tree of this graph. Its two edges have a common end, say x. Calling the other attachments y, t gives a plane drawing of $B + xy + xt$. □

Lemma 5.30. *The attachments of every Ψ_2-bridge B with nonempty interior can be labeled x, y, t so that $B+xy+xt$ is planar.*

Proof. Suppose some bridge B has no planar two-edge completion. Write x, y, t for its three attachments. Then

$$B + xy + xt, \qquad B + xy + yt, \qquad B + xt + yt$$

are all nonplanar.

Suppose first that B attaches at a, b and one of c, d, say c after interchanging c, d if necessary. By three-connectivity there is a c–d path in $G - \{a, b\}$. Since c is the only attachment of B left in this graph, the path avoids the interior of B. It also avoids the four chosen twins internally, since each has degree one in $G - \{a, b\}$. Contract the path into c and retain only B and the four twins with their incident edges. The two twins formerly adjacent to d now meet c, so all four have neighborhood $\{a, b, c\}$. This proper minor of G is toroidal. Lemma 4.18 therefore gives a planar two-edge completion of B, a contradiction.

Thus B attaches at $\{a, c, d\}$ or $\{b, c, d\}$. If every other component outside Ψ_2 had the same attachment triple, the omitted vertex, respectively b or a, would have only the four chosen twins as neighbors, contrary to Lemma 5.5. Retain Ψ_2, B, and one component with a different triple contracted to a vertex v with its three incident edges. Together B and the star at v meet all four vertices a, b, c, d.

Suppose the retained graph has a torus embedding. Since B joins c, d, a genus-one regular neighborhood of Ψ_2 would give B a planar two-edge completion by Lemma 5.29. Its regular neighborhood therefore has genus zero. In this case each boundary walk contains at most two distinct attachment labels [Supp. R4]. Both B and the star at v have three attachments, so each requires a region with at least two boundary components. By Lemma 2.1 they lie in the same annulus. Since together they meet all four attachments, the boundary sequences are $ac \mid bd$ or $ad \mid bc$ [Supp. R4].

By Lemma 2.4, B and the star at v have a plane drawing with ac, bd added in the first annulus or ad, bc in the second. The star joins three of a, b, c, d, and the added pair joins the fourth to one of these three. Within the star and the added pair, keep a tree joining the three attachments of B. Prune any leaves outside that triple, then contract edges with an end outside the triple until only the three attachments remain, keeping them distinct. Its two remaining edges give a planar two-edge completion of B, a contradiction.

Thus the retained graph is a nontoroidal minor of G, so minimality makes it G. An allowed reduction of B keeps its interior connected and adjacent to the same three attachments. If all three two-edge completions remained nonplanar, the same argument with the reduced bridge and the star at v would again give a nontoroidal minor of G. Hence no proper allowed reduction of B keeps all three completions nonplanar.

Suppose B is planar and that $B + xy$, $B + xt$ and $B + yt$ are all planar. Put $K = B + xy + xt + yt$. As established at the beginning of this subsection, K is three-connected and has no K_5 minor, and it contains the nonplanar graph $B + xy + xt$. By Lemma 3.8, there is a three-cut $X \neq \{x, y, t\}$ for which $K - X$ has at least three components, and two interior vertices have neighborhood X in G.

If X contains two attachments, call them x, y. By Lemma 3.8(vi), adding xy within X preserves the planarity of $B + xt$. This makes $B + xy + xt$ planar, a contradiction.

If X contains just one attachment, call it x. Suppose y, t lie in the same component D of $B - X$. The only added edge surviving in $K - X$ is yt, so D is also a component of $K - X$ and has neighborhood X in K. Every added edge leaving D ends at x. Thus D already has neighbors at both vertices of $X - \{x\}$ in B, and adding xy joins it to x. Contracting D now gives, with the two single vertices adjacent to all of X, a $K_{3,3}$ minor of the planar graph $B + xy$.

If instead y, t lie in different components of $B - X$, then xy and xt are added to different bridges on X. The bridge containing xy in $B + xy + xt$ is the same as in the planar graph $B + xy$, and the bridge containing xt is the same as in the planar graph $B + xt$. By Lemma 3.8(vi), each of these bridges has two attachments and stays planar when they are joined. The same lemma then makes $B + xy + xt$ planar, again a contradiction.

It follows that X lies entirely in the interior of B. The two vertices with neighborhood X give a twin triple disjoint from $\{a, b, c\}$, contrary to the hypothesis on G. Thus when B is planar, we may label its attachments so that $B + xy$ is nonplanar.

If B is nonplanar, it has no K_5 minor, so choose a $K_{3,3}$ subdivision $H \subseteq B$ and put $E = \varnothing$. If B is planar, put $E = \{xy\}$. Both $B + xy$ and $B + xt + yt$ are nonplanar. Since the triangle completion of B is three-connected and has no K_5 minor, Lemma 3.16 gives a $K_{3,3}$ subdivision $H \subseteq B + xy$ using xy for which $H - xy + xt + yt$ is nonplanar.

In either case apply Lemma 3.17 with attachments x, y, t and added edges E. It gives an allowed minor B' with at most eight interior vertices and a shortened subdivision $H' \subseteq B' + E$. When $E = \varnothing$, all three two-edge completions contain H'. When $E = \{xy\}$, the two containing xy contain H'. For the third, Lemma 2.6 gives the criterion for $H - xy + xt + yt$ to be nonplanar. The vertex t must lie off the cycle formed by the four branch paths avoiding the corner ends of the branch path through xy. Shortening preserves this position, including when t is outside H. Thus $B' + xt + yt$ contains the nonplanar graph $H' - xy + xt + yt$. In both cases all three completions remain nonplanar, so minimality gives $B' = B$.

Construction 3.18 lists these bridge forms with $E = \varnothing$ or $E = \{xy\}$, over all choices of x, y. Keep the forms whose three two-edge completions are nonplanar and combine each with Ψ_2 and the star at v. There are two choices for the attachment triple of B and three remaining choices for $N(v)$. These give the 5,934 graphs checked in [Supp. E18]. Each has a cut of at most two vertices, a K_5 minor, or a proper nontoroidal minor. Each possibility contradicts a defining property of G. □

5.6 Possible forms and combinations of bridges with three attachments

By Lemma 5.30, label the attachments of each bridge B by x, y, t so that $B + xy + xt$ is planar. Proposition 3.11 makes $B - x$ a chain of edges and four-cycles. We show that minimality leaves eight possible bridge forms and

then determine their possible combinations. With at least one component having more than one vertex, the only obstruction that remains is $\mathfrak{T}_{15}$. The next lemma lets us extend a drawing of a shortened bridge to a drawing of the original bridge.

Lemma 5.31. *Let B satisfy the hypotheses of Proposition 3.11, with attachments x, y, t, and let C be a four-cycle of $B - x$ containing neither y nor t. Let B' be a shortening of B, drawn in a plane graph K, possibly with its attachments split into copies. Suppose a connected subgraph H of K contains all these copies and is disjoint from the interior of B', and no edges of K outside B' meet its interior.*

If B' is obtained by either of the following operations:

(i) *Contract the path of one or two edges joining C to the next four-cycle. If the path has length two, first delete the edge from its middle vertex to x.*

(ii) *For an end path y, w, s or t, w, s with $s \in V(C)$, delete wx and contract ws.*

then the drawing of B' can be extended to a drawing of B, keeping the same attachment copies and leaving the rest of the drawing of K fixed.

Proof. We first put back the contracted path. Let s be the vertex created by the contraction, and write $C = spuqs$. Since H is connected and disjoint from C, it lies on one side of C. Every edge at s outside C begins a path to H that does not meet C again: through the next four-cycle in (i), or along the remaining edge to y or t in (ii). These edges therefore all leave C on the side containing H, so sp, sq are consecutive around s. We can therefore split s into two vertices s', s'', keeping the two edges of C at s' and the other edges at s''. Restore the contracted edge or two-edge path between them inside a small disk around s, without crossings. If an edge to x was deleted, it remains to put that edge back. Let x_p be the copy of x adjacent to p. The restored path borders the face beside $s'p$ on the side containing H. Since p has degree three and px_p lies on this same side of C, the boundary of this face continues along px_p. We can draw the deleted edge in this face, close to the path $s'px_p$, starting at the restored middle vertex in (i), or at $s'' = w$ in (ii). This restores B without crossings and fixes the rest of the drawing. □

We now apply Lemma 5.31 in the disk and annulus tests to bound the number of four-cycles and the lengths of the paths joining them.

Lemma 5.32. *Suppose no choice of twin pairs gives a bridge with four attachments. Let B be a Ψ_2-bridge with nonempty interior and attachments x, y, t, labeled so that $B + xy + xt$ is planar. Then B has one of the following eight forms, shown in Figure 15. The end lengths count the edges from y, t to the nearest four-cycle in $B - x$. The edges incident with x are those specified in Proposition 3.11. Each form has at most seven interior vertices.*

Proof. By Proposition 3.11, the blocks of $B - x$ form a chain of edges and four-cycles, with no three consecutive edge blocks. Two four-cycles cannot share a vertex v. Their degree-three vertices would give twin triples $\{x, u, v\}$ and $\{x, v, w\}$, while v is interior to B and has degree four in G, contrary to Lemma 5.5. Thus consecutive cycles are joined by a path of length one or two, and each end length is at most two. Lemma 5.5, also with a, b interchanged, gives $\deg(a), \deg(b) > 4$. Since c, d have degree at least three, the additions to Ψ_2 meet all four vertices. As B has only three attachments, there is another bridge or the edge cd.

Suppose a shortening in (i) or (ii) of Lemma 5.31 is possible. It keeps the interior connected and preserves all three attachments, giving a proper minor of G. Take a torus embedding of this minor. The subgraph Ψ_2 and every bridge's attachments are unchanged. If a closed regular neighborhood of Ψ_2 has genus zero, no boundary walk contains three attachments or both c, d [Supp. R4]. The shortened bridge therefore cannot lie in a disk or a torus with one open disk removed, since each has only one boundary walk. Lemma 2.1 then forces an annulus, containing every bridge with three attachments and cd if present. Thus the disk and annulus tests apply. For a disk containing the shortened bridge, take $H = C_F$, the boundary cycle in its planar test graph. For an annulus, the additions still meet all four vertices, so its boundary sequences are $ac \mid bd$ or $ad \mid bc$. Take H to consist of the two added edges of the annulus test together with cd or another bridge. The edge cd joins the two added edges, and any bridge with three attachments meets both. Thus H is connected, contains all four attachment appearances, and avoids the interior of the shortened bridge. Lemma 5.31 therefore restores B in the planar test graph, keeping the other edges and their attachment appearances fixed. Lemma 2.3 or Lemma 2.4 then gives a torus embedding of G, a contradiction.

Consequently, a four-cycle avoiding both y, t cannot occur together with another four-cycle. There are at most two cycles, and if there are two, they contain y, t respectively. Their end lengths are therefore $(0, 0)$, and the joining path has length one or two. With one cycle, end lengths $(0, 0)$ would disconnect the interior of B. If both end lengths are positive, the cycle avoids y, t, so a length-two end could be shortened and restored. Both lengths must therefore be one. This leaves $(0, 1), (0, 2), (1, 0), (2, 0), (1, 1)$. With no cycle, Proposition 3.11 gives the star. These

are the eight stated forms. The largest has two cycles containing y, t and one vertex between them, giving seven interior vertices. □

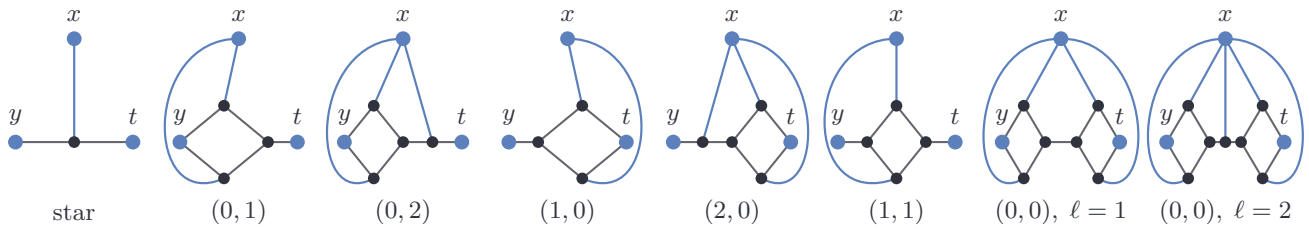


Figure 15: The eight bridge forms in Lemma 5.32. The attachments and the edges incident with x are blue.

Proposition 5.33. *If some component outside Ψ_2 is not a singleton, then $G \cong \mathfrak{T}_{15}$.*

Proof. Suppose three distinct twin triples contain the same two vertices s, t, and choose twins p_i, q_i with $N(p_i) = N(q_i) = \{s, t, x_i\}$ for $i = 1, 2, 3$. These six vertices are distinct and lie outside $\{s, t, x_1, x_2, x_3\}$ by Lemma 3.4(b). Since $G - \{s, t\}$ is connected, choose a minimal tree H containing x_1, x_2, x_3. Every vertex of degree one in H belongs to $\{x_1, x_2, x_3\}$, so H avoids the six twins, which have degree one in $G - \{s, t\}$. The tree H has at least two vertices of degree one, all belonging to $\{x_1, x_2, x_3\}$. Permute the indices in the labels x_i, p_i, q_i so that x_1, x_2 have degree one in H, and let D be the component of $G - \{s, t, x_1, x_2\}$ containing $H - \{x_1, x_2\}$. The tree gives D neighbors at x_1, x_2, while $p_3, q_3 \in V(D)$ give neighbors at s, t. Since the other four twins are isolated after these deletions, choosing p_1, q_1, p_2, q_2 as the four twins gives a bridge with attachments $\{s, t, x_1, x_2\}$. This contradicts the assumption that no choice of twin pairs gives a bridge with four attachments.

By Lemmas 5.28 and 5.32, G consists of Ψ_2, one to four bridges of the eight listed forms, and possibly cd, with at least one bridge having more than one interior vertex. Assigning each form's attachments to three of a, b, c, d in every possible order gives 76 labeled bridges up to isomorphisms fixing the attachments. Combining these with disjoint interiors under the preceding conditions, allowing repetitions and including or omitting cd, gives 3,163,020 combinations. We discard combinations with two disjoint twin triples, three distinct twin triples containing the same two vertices, or a bridge with four attachments for some choice of the four twins. These exclusions remain valid when further bridges or cd are added, since only a, b, c, d gain neighbors and none is a degree-three vertex with a twin. Indeed, a, b have degree at least four, while a twin of c would have to be a or b to be adjacent to both p, q, and the same argument applies to d. Thus existing degree-three twins keep their neighborhoods. If adding cd puts an edge inside one of these neighborhoods, Lemma 3.4(a) excludes the graph, and otherwise the twin triples remain. A component with four attachments also retains those attachments when it becomes part of a larger component. We further discard any combination containing a smaller nontoroidal combination, since it has a proper nontoroidal minor. After these exclusions, the checks in [Supp. E19] give a cut of at most two vertices, a torus embedding, or a proper nontoroidal minor for every combination except four. These four graphs are all isomorphic to $\mathfrak{T}_{15}$. □

Proof of Theorem 5.1. If some choice of twin pairs gives a Ψ_2-bridge with four attachments, Lemma 5.2 leaves $\mathfrak{T}_2$ or $(n_c, n_d) = (2, 2), (3, 2), (3, 3)$. Lemma 5.6 excludes $(3, 3)$. Proposition 5.9, together with Lemma 5.10 and Propositions 5.16–5.27, cover the rest. If no choice gives such a bridge, Lemma 5.28 and Proposition 5.33 give the remaining graphs. The graphs $\mathfrak{T}_{48}, \mathfrak{T}_{66}$ have disjoint twin triples and belong to Theorem 4.1. □

6 Two twin triples with one common vertex

Let G be a three-connected obstruction with at least two distinct twin triples, every two of which have exactly one vertex in common. Choose twins p, q with neighborhood $A_1 = \{a, b, c\}$ and twins r, z with neighborhood $A_2 = \{a, d, e\}$. Write J_1, J_2 for their twin subgraphs and put $\Psi_3 = J_1 \cup J_2$, as in Figure 16. Let n_c, n_e count the vertices with neighborhoods A_1, A_2, respectively.

Theorem 6.1. *A graph is a three-connected obstruction with at least two distinct twin triples, every two of which have exactly one vertex in common, if and only if it is isomorphic to one of*

$$\mathfrak{T}_7, \mathfrak{T}_{14}, \mathfrak{T}_{20}, \mathfrak{T}_{23}, \mathfrak{T}_{29}, \mathfrak{T}_{33}, \mathfrak{T}_{35}, \mathfrak{T}_{37}, \mathfrak{T}_{41}, \mathfrak{T}_{49}, \mathfrak{T}_{54}, \mathfrak{T}_{55}, \mathfrak{T}_{59}, \mathfrak{T}_{70}.$$

Every component of $G - V(\Psi_3)$ has at least three neighbors, all in $A_1 \cup A_2$, since G is three-connected and the twins have degree three. The sets A_1, A_2 are independent by Lemma 3.4(a), so the Ψ_3-bridges with empty

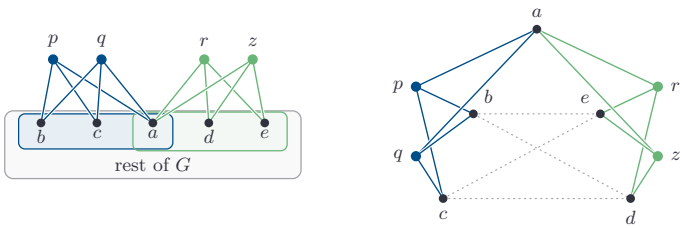


Figure 16: Left, two twin triples meeting only at a. Right, the graph Ψ_3, with the possible cross edges dotted.

interior are edges joining $\{b, c\}$ to $\{d, e\}$, which we call *cross edges*. We first derive the embedding tests, then consider bridges with five attachments, four attachments excluding a, four attachments including a, and finally three attachments. The next lemma supplies the minors of G used throughout this section.

Lemma 6.2. *Let U be a union of Ψ_3-bridges. The graph obtained from U by adding the vertices a, b, c, d, e and the edges of the triangles abc and ade is a minor of G, and so has no K_5 minor.*

Proof. Contract pb, qc, rd, ze and delete the bridges outside U. □

6.1 Faces and embedding tests of Ψ_3

Lemma 6.3. *In a torus embedding of Ψ_3, let $\mathcal{N}$ be a closed regular neighborhood of Ψ_3. If $\mathcal{N}$ has genus one and a face meets b, c, d, e, that face is a disk with attachment sequence a, b, a, d, a, c, a, e, up to cyclic shift and reversal, and the other two faces have attachment sequences bc, de. If $\mathcal{N}$ has genus zero, any face meeting b, c, d, e is an annulus with attachment sequences $bc \mid de$, and the other three faces are disks with attachment sequences $axay$, ax' and ay', where $\{x, x'\} = \{b, c\}$ and $\{y, y'\} = \{d, e\}$.*

Proof. Ψ_3 has 96 rotation systems. Their facial walks are recorded in [Supp. R6]. No rotation system of genus zero has a face containing all four of b, c, d, e in its boundary walk. The only pair of facial walks in any rotation system of genus zero whose union contains all four has attachment sequences bc, de. Thus, by Lemma 2.1, a face meeting all four in a torus embedding with this rotation must be an annulus with attachment sequences $bc \mid de$. The other three facial walks of such a rotation have attachment sequences $axay$, ax' and ay', and bound disks by the same lemma. Of the genus-one rotations, the only facial attachment sequence containing all four is $abadacae$, up to cyclic shift and reversal. The other two faces have attachment sequences bc, de. □

The disk boundary in Figure 17 follows the star $ab + ac + ad + ae$, with ab, ac alternating with ad, ae at a. The next lemma determines when a planar graph containing this star has an embedding with that order.

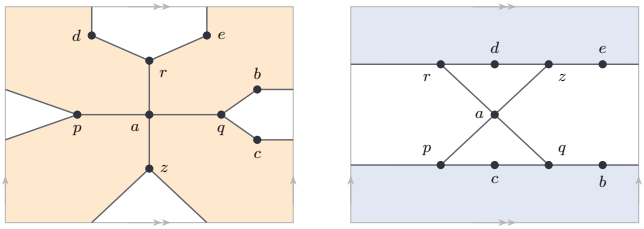


Figure 17: Embeddings of Ψ_3 in Lemma 6.3, with a disk face shaded on the left and an annular face on the right.

Lemma 6.4. *Let H be planar, let $H - a$ be connected, and let b, c, d, e be distinct neighbors of a. There is a plane embedding of H in which ab, ac alternate with ad, ae around a if and only if $H - a$ has no vertex-disjoint b–c and d–e paths.*

Proof. Suppose $H - a$ has vertex-disjoint b–c and d–e paths. The b–c path, together with ab, ac, forms a cycle. In a plane drawing of H in which ab, ac alternate with ad, ae around a, one of d, e lies in the interior of this cycle and the other does not. Since it avoids a, the d–e path must cross an edge of the cycle or meet the b–c path at b, c or one of its interior vertices. This contradicts planarity or the vertex-disjointness of the two paths.

Now suppose no such paths exist. Choose a plane embedding of H. If the four edges already alternate, we are done. Otherwise, after possibly interchanging b, c or d, e, their cyclic order at a is ab, ac, ad, ae. Subdivide every edge at a and delete a, writing b', c', d', e' for the subdivision vertices on the four specified edges. All subdivision

vertices lie on a common face, which we take as the outer face. Draw the resulting graph in a disk meeting its boundary only at e', b', c', d' in this order. Deleting these four vertices leaves a connected graph, since $H - a$ is connected. Lemma 2.9 gives boundary order e', b', d', c', keeping all subdivision vertices on the outer face, since its disjoint-path alternative would give the excluded paths in $H - a$. Insert a in that face and join it to every subdivision vertex. Replace each resulting two-edge path from a to an original neighbor by a single edge. This gives the required embedding of H. □

Lemma 6.5. *Let G be a three-connected obstruction with twin triples $A_1 = \{a, b, c\}$ and $A_2 = \{a, d, e\}$ satisfying $A_1 \cap A_2 = \{a\}$. Let n_c and n_e denote the numbers of vertices of G with neighborhoods A_1 and A_2, respectively. Then at least one of these twin triples is the neighborhood of exactly two vertices*

$$\min(n_c, n_e) = 2.$$

Proof. Otherwise, three vertices with neighborhood $\{a, b, c\}$ and three with neighborhood $\{a, d, e\}$ give a subgraph H of G isomorphic to $\mathfrak{T}_8$. Since H has cut vertex a and G is three-connected, $H \neq G$. Thus H is a proper nontoroidal subgraph of G, contradicting minimality. □

Lemma 6.6. *Let B be a Ψ_3-bridge whose attachment set contains $\{b, c, d, e\}$. Every other Ψ_3-bridge with nonempty interior has attachment set $\{a, b, c\}$, $\{a, d, e\}$ or $\{a, x, y\}$, where $x \in \{b, c\}$ and $y \in \{d, e\}$. The distinct pairs xy for which $xy \in E(G)$ or a bridge has attachment set $\{a, x, y\}$ form a matching.*

Proof. Contract J_1, J_2 to the triangles abc, ade and the interior of B to w. If another bridge has at least three attachments among b, c, d, e, exchange the two twin triples if necessary so that it meets both b, c, then interchange d, e if necessary so that it also meets d. Contract its interior to u, then contract ae into a and du into d. This gives a K_5 on a, b, c, d, w, a contradiction. Every other bridge with nonempty interior therefore has at most two attachments among b, c, d, e. Three-connectivity gives at least three attachments altogether, so its attachment set is one of the stated triples. Contracting the interior of a bridge on $\{a, x, y\}$ into x supplies the edge xy. If two distinct pairs share an end, label them bd, be and obtain either missing edge by the corresponding bridge contraction. With the triangles and w as above, contracting ac into a gives a K_5 on a, b, d, e, w, again a contradiction. Hence the pairs form a matching. □

6.2 A bridge with five attachments

This subsection shows that a bridge with five attachments gives $\mathfrak{T}_{20}$ or $\mathfrak{T}_{23}$. Corollary 6.7 gives the embedding test for such a bridge, and Proposition 6.8 reduces the bridge and enumerates the possible forms.

Suppose a Ψ_3-bridge B has all five attachments. Lemma 6.3 tells us it must lie in a disk with attachment sequence a, b, a, d, a, c, a, e. Put $H = G - \{p, q, r, z\} + ab + ac + ad + ae$. Contracting ap, ar and deleting q, z gives H, so H has no K_5 minor.

Corollary 6.7. *Let F be a disk face of a torus embedding of Ψ_3 with attachment sequence abadacae, and let U be a union of Ψ_3-bridges, including a, b, c, d, e. Then U can be drawn in F if and only if $U + ab + ac + ad + ae$ has a plane embedding in which ab, ac alternate with ad, ae at a. In particular, if a Ψ_3-bridge B has all five attachments, then G is toroidal if and only if*

$$H = G - \{p, q, r, z\} + ab + ac + ad + ae$$

is planar and $H - a$ has no vertex-disjoint b–c and d–e paths.

Proof. Put $S = ab + ac + ad + ae$. Its plane embedding with cyclic order ab, ad, ac, ae at a has facial walk $abadacae$. Placing U in this face gives the required plane embedding of $U + S$, and cutting such an embedding along S gives a drawing of U in F.

If B has all five attachments, Lemma 6.3 places it in the disk in every torus embedding of G, since the annulus does not meet a. Bridges with nonempty interior have at least three attachments, and every cross edge joins $\{b, c\}$ to $\{d, e\}$, so none can lie in the other two faces with attachment sequences bc, de. Thus, by the first claim and Figure 17, G is toroidal if and only if H has the stated alternating plane embedding. Each bridge with a deleted is connected and meets $B - a$, so $H - a$ is connected. Lemma 6.4 now gives the stated test. □

Proposition 6.8. *If, for some choice of the four twins defining Ψ_3, a Ψ_3-bridge has all five attachments, then G is isomorphic to $\mathfrak{T}_{20}$ or $\mathfrak{T}_{23}$.*

Proof. Let B be such a bridge. By Lemma 6.6, every other component has neighborhood $\{a,b,c\}$, $\{a,d,e\}$ or $\{a,x,y\}$, where $x \in \{b,c\}$ and $y \in \{d,e\}$. If a component has neighborhood $\{a,b,c\}$, keep only Ψ_3 and the two bridges, contract their interiors to vertices, and delete the edges from the vertex representing B to b,c. This gives $\mathfrak{T}_8$ [Supp. W9], a proper nontoroidal minor since it has a cut vertex and G is three-connected. The same argument excludes $\{a,d,e\}$.

The distinct pairs supplied by cross edges or the remaining components form a matching by Lemma 6.6. If there are two, label them bd, ce and obtain either missing edge by the corresponding bridge contraction. Contract J_1, J_2 to the triangles abc, ade and the interior of B to w. Contracting bd then gives a K_5, a contradiction. Thus, after relabeling within $\{b,c\}$ and $\{d,e\}$, bd is the only possible cross edge and every other component has neighborhood $\{a,b,d\}$. If there are two such components, contracting them separately gives two vertices with this neighborhood. A bridge B' with nonplanar triangle completion gives the same configuration. Indeed, Lemmas 3.1 and 3.13 apply using the outside paths apb, ard and a b–d path through the interior of B, and contracting the attachment edge at s in $T(s)$ gives the required vertices. In either case, contract the interior of B to w. Keep only Ψ_3, the five edges from w to a,b,c,d,e, and the two vertices with neighborhood $\{a,b,d\}$ together with their incident edges. This gives $\mathfrak{T}_{22}$ [Supp. W10]. Its twin triples $\{a,b,c\}, \{a,b,d\}$ meet twice, so it is a proper nontoroidal minor, a contradiction. Hence there is at most one other component, and its bridge has planar triangle completion.

Suppose $H - a$ contains vertex-disjoint paths P from b to c and Q from d to e. After deleting a, the possible other bridge attaches only at b,d. A path passing through its interior must therefore contain both b and d, and a path using bd also contains both. Neither P nor Q can do this, since b belongs to P and d belongs to Q. Thus every internal vertex of P and every internal vertex of Q lies in the interior of B. Both paths have internal vertices because $bc, de \notin E(G)$. Start with the internal vertices of P in one set and those of Q in another. Repeatedly add a vertex of the interior of B outside the two sets to a set containing one of its neighbors. Since the interior of B is connected, this can continue until every interior vertex belongs to one of the two sets. The sets remain disjoint and connected, and connectedness of the whole interior gives an edge between them. Contract them to u, v, respectively. The paths P, Q give the edges ub, uc, vd, ve, and the edge between the sets gives uv. Also, a has an interior neighbor in B, so at least one of au, av is present. Keep one of these edges together with ub, uc, uv, vd, ve and Ψ_3. This gives $\mathfrak{T}_6$ [Supp. W11]. Deleting a and the vertex of $\{u,v\}$ adjacent to a disconnects this minor, so it is proper since G is three-connected, a contradiction.

Thus no such paths exist, and Corollary 6.7 forces H to be nonplanar. If another bridge B' remains, its triangle completion is planar. Lemma 2.5(ii), applied on $\{a,b,d\}$ to B' and $B + ac + ae$, therefore makes $B + ab + ac + ad + ae + bd$ nonplanar. Contract the interior of B' into b and delete the resulting edge ab. This gives the proper minor $(\Psi_3 \cup B) + bd$, which is nontoroidal by Corollary 6.7, a contradiction. Hence $G = \Psi_3 \cup B$ or $(\Psi_3 \cup B) + bd$.

The nonplanar graph H has no K_5 minor. If bd is present, Corollary 6.7 applied to the toroidal graph $G - bd$ makes $H - bd$ planar, so every $K_{3,3}$ subdivision of H uses bd. Replacing B by an allowed minor B' preserves the connected interior and all five attachments, so by Corollary 6.7 it gives a nontoroidal graph whenever the corresponding completion of B' is nonplanar. Corollary 3.19, with the five attachments and the fixed planar graph on them consisting of ab, ac, ad, ae, together with bd when present, gives 217,376 candidates. All except $\mathfrak{T}_{20}, \mathfrak{T}_{23}$ have a K_5 minor, a cut of at most two vertices, a proper nontoroidal minor, or two twin triples meeting in zero or two vertices [Supp. E20]. □

6.3 A bridge with four attachments, not including a

Assume that no choice of the four twins gives a bridge with five attachments. Choose a Ψ_3-bridge B with attachment set $\{b,c,d,e\}$. In any torus embedding of $\Psi_3 \cup B$, Lemma 6.3 places B in the disk or annulus of Figure 17. By Lemma 6.6, every other Ψ_3-bridge with nonempty interior has three attachments, including a. Let B_1 be their union, and let E be the set of cross edges. Put

$$U = G - \{p,q,r,z\}, \qquad H = U + ab + ac + ad + ae.$$

Thus U consists of B, B_1, the vertex a and the edges in E. The next lemma gives an embedding test for each placement of B.

Lemma 6.9. *The graph $\Psi_3 \cup U$ is toroidal if and only if at least one of the following holds.*

(1) *H has a plane embedding in which ab, ac alternate with ad, ae at a.*

(2) *For some $x \in \{b,c\}$ and $y \in \{d,e\}$, every bridge in B_1 has attachment set $\{a,x,y\}$, and there is $E_0 \subseteq E \cap \{xy\}$ such that both of the following graphs are planar*

$$B + bc + de + (E \setminus E_0), \qquad B_1 + ax + ay + E_0.$$

Proof. In any torus embedding, Lemma 6.3 places B in the disk with attachment sequence *abadacae* or in the annulus $bc \mid de$. In the disk case, the other two faces meet only b, c or only d, e, so all bridges lie in the disk. Corollary 6.7 makes this placement possible exactly when (1) holds. In the annular case, every bridge in B_1 has three attachments including a, so by Lemma 6.3 it must lie in the disk with attachment sequence $axay$, for some $x \in \{b, c\}$ and $y \in \{d, e\}$. Thus every such bridge has attachment set $\{a, x, y\}$. Only the cross edge xy can also lie in this disk. Let E_0 be the set of cross edges placed there. By Lemma 2.4, B and the remaining cross edges fit in the annulus exactly when $B + bc + de + (E \setminus E_0)$ is planar. The path xay has facial walk $axay$. Inserting a drawing into this face, or cutting a plane embedding along the path, shows that the disk drawing exists exactly when $B_1 + ax + ay + E_0$ is planar. These drawings occupy distinct faces of the same embedding of Ψ_3. Interchanging b, c and interchanging d, e gives every choice of x, y, so the annular placement is possible exactly when (2) holds. □

The next lemma shows that nonplanarity of $B + bc + de$ implies nonplanarity of $B + ab + ac + ad + ae$ when their union has no K_5 minor.

Lemma 6.10. *Let F be a connected graph with distinct vertices b, c, d, e, and put $H = F + ab + ac + ad + ae$, where a is a new vertex. If H is planar and $H + bc + de$ has no K_5 minor, then $F + bc + de$ is planar.*

Proof. Suppose $F + bc + de$ is nonplanar. If ab, ac do not alternate with ad, ae in some plane embedding of H, draw bc, de alongside bac, dae and delete a. This gives a plane embedding of $F + bc + de$, a contradiction. Thus neither of the other two pairings alternates at a in any plane embedding of H. Since $H - a = F$ is connected, Lemma 6.4 gives disjoint b–d and c–e paths P, Q in F, and disjoint b–e and c–d paths.

On each of the latter two paths, take a shortest subpath joining P to Q. These subpaths are disjoint and meet P, Q only at their ends. Together with the portions of P, Q between their ends, they form a cycle C. Contract the remaining portions of P, Q onto C. In the plane embedding of H, the vertices now occur on C in the order of their edges at a, namely b, d, c, e up to reversal. Shorten the four arcs of C to single edges. Together with bc, de and the four edges at a, this gives a K_5 minor of $H + bc + de$, a contradiction. □

We next restrict the other components and settle the case where $B + bc + de$ is nonplanar.

Lemma 6.11. *After relabeling within $\{b, c\}$ and $\{d, e\}$, every component of $G - V(\Psi_3)$ other than the interior of B has neighborhood $\{a, b, d\}$ or $\{a, c, e\}$, and every cross edge is bd or ce. If $B + bc + de$ is nonplanar, then $G \cong \mathfrak{T}_{54}$.*

Proof. If another component has neighborhood $\{a, b, c\}$, contract it to u and the interior of B to w. Keep only Ψ_3 and the edges at u, w, then delete bw. This gives a proper nontoroidal minor $\mathfrak{T}_3$ [Supp. W12], a contradiction. The case $\{a, d, e\}$ is symmetric. Lemma 6.6 now gives the stated neighborhoods and cross edges.

Put $H_2 = B + bc + de$ and suppose it is nonplanar. By Lemma 6.2, $B + ab + ac + ad + ae + bc + de$ is a minor of G. Thus H_2 has no K_5 minor. Let B' be an allowed minor of B with $B' + bc + de$ nonplanar. The graph $B' + ab + ac + ad + ae + bc + de$ is still a minor of G, so Lemma 6.10 makes $B' + ab + ac + ad + ae$ nonplanar. Thus $\Psi_3 \cup B'$ fails both tests of Lemma 6.9. Taking $B' = B$, minimality gives $G = \Psi_3 \cup B$, and Corollary 3.19, with attachments b, c, d, e and added edges bc, de, includes B among 26,633 forms. Every resulting graph other than $\mathfrak{T}_{54}$ is toroidal or has a K_5 minor or a proper nontoroidal minor [Supp. E21], so $G \cong \mathfrak{T}_{54}$. □

We may now assume $H_2 = B + bc + de$ is planar. If B_1 is empty, condition (2) of Lemma 6.9 forces $|E| = 2$. Let E' consist of E and the edges xy supplied by components with neighborhood $\{a, x, y\}$. Thus E' is nonempty, and Lemma 6.6 allows us to label it as $\{bd\}$ or $\{bd, ce\}$. Put

$$H_3 = B + ab + ac + ad + ae + bc + de + E'.$$

Lemma 6.12. *The graph H_3 is three-connected and has no K_5 minor. If it is nonplanar, then the interior of B contains twins p', q' with neighborhood $\{x, y, w\}$, where $w \in \mathrm{Int}(B)$ and xy is bd or ce. Moreover, $xy \notin E(G)$, and $B[\mathrm{Int}(B) \setminus \{p', q'\}]$ is connected and has neighbors at the other two vertices of $\{b, c, d, e\}$.*

Proof. By Lemma 6.2, contracting each other component with neighborhood $\{a, x, y\}$ into x gives H_3 as a minor of G, so it has no K_5 minor. Since the neighbors of each component of $G - V(H_3)$ form a triangle in H_3, any cut of H_3 also separates G. Thus H_3 is three-connected.

Suppose H_3 is nonplanar. The triangle abc excludes V_8, so Halin's theorem gives a three-cut X leaving at least three components, each with neighborhood X. If $a \in X$, each component contains a neighbor of a, but all four neighbors lie on the path $cbde$, which has at most two pieces after deleting the other two cut vertices. Hence $a \notin X$. All surviving attachments lie with a, leaving at least two components entirely inside B. These are components of $B - X$, so Lemma 3.3 gives exactly two single vertices p', q' with neighborhood X in G.

The twin triple X meets each of $\{a,b,c\}$ and $\{a,d,e\}$ exactly once. Thus $X = \{x,y,w\}$ with $x \in \{b,c\}$, $y \in \{d,e\}$ and $w \in \mathrm{Int}(B)$. In the interior of B, p', q' are leaves at w. Deleting them leaves the interior connected and preserves its neighbors at the other two attachments. Contract this remaining interior to w. If $xy = be$, contracting bp', eq', cw gives a K_5 on a,b,c,d,e. Interchanging b,d and c,e excludes cd as well. Hence xy is bd or ce, and $xy \notin E(G)$ since $N_G(p')$ is independent by Lemma 3.4(a). □

We now complete the case where a bridge has attachment set $\{b,c,d,e\}$.

Proposition 6.13. *If no choice of the four twins gives a bridge with five attachments, but some choice gives a bridge with attachment set $\{b,c,d,e\}$, then G is isomorphic to one of $\mathfrak{T}_{29}, \mathfrak{T}_{54}, \mathfrak{T}_{55}$.*

Proof. Lemma 6.11 handles nonplanar H_2, so assume H_2 is planar. For each other bridge on $\{a,x,y\}$, the paths apx, ary and an x–y path through the interior of B join its attachments with disjoint interiors outside the bridge. Its triangle completion is therefore a minor of G, and is three-connected by Lemma 3.1. If this completion is nonplanar, Lemma 3.13 gives an allowed minor $T(s)$ for some $s \in \{a,x,y\}$. Contracting its edge at s gives two vertices with neighborhood $\{a,x,y\}$. Suppose there are at least two other components and one bridge has nonplanar triangle completion. Interchange b,c and d,e simultaneously if necessary so that its attachments are $\{a,b,d\}$, and replace it by the two vertices just described. Contract another component to v and the interior of B to w, and delete the other bridges and cross edges. If $N(v) = \{a,b,d\}$, the three vertices make both tests nonplanar by Lemma 2.5(i). If $N(v) = \{a,c,e\}$, the two new vertices and the connected graph on c,e,v,w give three bridges adjacent to all of a,b,d in the disk test, making it nonplanar. The annular test fails because both attachment triples occur. This nontoroidal minor has a twin triple meeting $\{a,b,c\}$ twice, contradicting minimality. Thus, when at least two other components occur, all their triangle completions are planar. Lemma 3.2 makes them single vertices, with at most one of each neighborhood, since two would give twin triples meeting twice. Hence there are exactly two, say u,v, with neighborhoods $\{a,b,d\}$ and $\{a,c,e\}$. Their independent neighborhoods give $E = \varnothing$.

If B_1 is empty, then $E = \{bd, ce\}$. Condition (2) of Lemma 6.9 makes $H_2 + bd$ and $H_2 + ce$ nonplanar, so H_3 is nonplanar. Lemma 6.12 then makes bd or ce a nonedge of G, a contradiction. Suppose next that B_1 is one bridge. Interchange b,c and d,e simultaneously if necessary so that its attachments are $\{a,b,d\}$, and put $H_4 = B_1 + ab + ad$. If $E = \varnothing$, condition (2) makes H_4 nonplanar. Contracting the interior of B leaves H_4 unchanged, so the graph in condition (1) still contains the nonplanar subgraph H_4, while condition (2) requires H_4 to be planar. Minimality therefore makes the interior of B a single vertex. If B_1 is replaced by an allowed minor B_1' with $B_1' + ab + ad$ nonplanar, neither condition (1) nor condition (2) can hold. Thus Corollary 3.19, with attachments a,b,d and added edges ab, ad, gives the 349 forms in [Supp. E22]. Every resulting graph has a K_5 minor, a cut of at most two vertices, or a proper nontoroidal minor.

Now let $E \neq \varnothing$. Then H_4 is planar, since otherwise deleting E gives a proper nontoroidal minor. If $ce \in E$, then H_3 is nonplanar. Otherwise $H_2 + E$ and H_4 would satisfy condition (2) with $E_0 = \varnothing$. Lemma 6.12 gives twins p', q' with neighborhood $\{b,d,w\}$ and excludes bd as an edge. Contract the remaining interior of B to w, keeping only cw, ew and the six edges at p', q' in B. Contract the interior of B_1 to u. Together with Ψ_3 and ce, this gives $\mathfrak{T}_{29}$ [Supp. W13], so $G \cong \mathfrak{T}_{29}$. It remains to consider $E = \{bd\}$. The two choices of E_0 in condition (2) make $H_2 + bd$ and $H_4 + bd$ nonplanar. The latter is the triangle completion of B_1, so the preceding reduction gives an allowed minor $T(s)$ of B_1. Its triangle completion is nonplanar by Lemma 3.10, keeping the disk test and the annular choice $E_0 = \{bd\}$ failing. The choice $E_0 = \varnothing$ still fails because $H_2 + bd$ is nonplanar. Minimality gives $B_1 = T(s)$. For $s = a$, its twins have adjacent neighbors b,d, contrary to Lemma 3.4(a). For $s = b$ or $s = d$, their neighborhood meets $\{a,d,e\}$ or $\{a,b,c\}$, respectively, twice, again a contradiction.

Finally suppose there are two other components. They are the vertices u,v described above, and $E = \varnothing$. If H is planar, the failed disk test and Lemma 6.4 give vertex-disjoint b–c and d–e paths in the connected graph $H - a$. Neither can use u,v without meeting an end of the other path, so their nonempty interiors lie in the interior of B. Extend them to two connected sets partitioning the interior of B and contract these sets to y,t, keeping by, cy, yt, dt, et. Together with Ψ_3, u, v, this gives $\mathfrak{T}_{55}$ [Supp. W14], so $G \cong \mathfrak{T}_{55}$. If H is nonplanar, then H_3 is nonplanar as well. Otherwise each triangle abd, ace bounds a face in a plane drawing of H_3, since deleting its vertices leaves the connected interior of B and the other two attachments. Inserting u,v in these distinct faces and deleting bc, de, bd, ce gives a plane drawing of H, a contradiction. After simultaneously interchanging b,c, d,e and u,v if necessary, Lemma 6.12 gives twins p', q' with neighborhood $\{b,d,w\}$. Reduce B as in the case $ce \in E$, then contract cv and delete the new edge ac. This gives the same $\mathfrak{T}_{29}$ minor. It is proper because cv was contracted, contradicting minimality. □

6.4 A bridge with four attachments, including a

Assume that no choice of the four twins gives a bridge with five attachments or four attachments excluding their common vertex. Choose the four twins and a bridge B with four attachments to maximize $|\operatorname{Int}(B)|$, and call its attachments $\{a, c, d, e\}$.

Lemma 6.14. *After interchanging d, e if needed, every cross edge is bd or cd, and every component of $G - V(\Psi_3)$ other than the interior of B is a single vertex with neighborhood $\{a, d, e\}$, $\{a, b, d\}$ or $\{a, c, d\}$. Also, there is a b–d path whose interior avoids $V(\Psi_3) \cup \operatorname{Int}(B)$.*

Proof. Contract J_1, J_2 to the triangles abc, ade and the interior of B to w. Contracting bc into b gives all edges of K_5 on a, b, d, e, w except possibly bd, be. If cross edges or other components join $\{b, c\}$ to both d, e, contracting those components into b or c first supplies both missing edges, a contradiction. Thus, after interchanging d, e if needed, every cross edge is bd or cd, and the other components have neighborhoods among

$$\{a, b, c\}, \quad \{a, d, e\}, \quad \{a, b, d\}, \quad \{a, c, d\}, \quad \{b, c, d\}, \quad \{a, b, c, d\}.$$

For neighborhoods $\{a, b, c\}$, $\{b, c, d\}$ or $\{a, b, c, d\}$, contract the component to u and the interior of B to w, and delete the other bridges and cross edges. Deleting aw, cw, or au, cw, respectively, gives a proper $\mathfrak{T}_3$ minor [Supp. W15]. Hence only $\{a, d, e\}, \{a, b, d\}, \{a, c, d\}$ remain. Since b has only the neighbors p, q in Ψ_3, three-connectivity forces bd or a component with neighborhood $\{a, b, d\}$. Either contains the required path of the lemma statement.

Suppose some other bridge B_1 with nonempty interior has nonplanar triangle completion. The completion is three-connected by Lemma 3.1. If its attachments are $\{a, d, e\}$ or $\{a, c, d\}$, apply Lemma 3.13 using ard, aze and a d–e path through the interior of B, or apc, ard and a c–d path through the interior of B, respectively. This gives an allowed minor $T(s)$. Contracting its edge at s gives two vertices with the same attachment triple. The b–d path found above is either bd or has its interior in a component with neighborhood $\{a, b, d\}$, so its interior is disjoint from B_1. Contract this path into d and the interior of B to w. Keep only the edges of Ψ_3 after identifying b with d, the edges at the two new vertices, and aw, cw, dw, ew. Delete any isolated vertices. The resulting graph is $\mathfrak{T}_2$ [Supp. W16], with c, e interchanged in the second case. This minor is proper because b, d were identified, a contradiction. Thus B_1 has attachments $\{a, b, d\}$. By Lemma 6.2, its triangle completion is a minor of G, obtained by contracting the interior of B into d and c into d.

Suppose $B_1 + ab + ad$ is planar. Lemma 3.9 gives twins p', q' whose neighborhood X contains a. Since X meets each of $\{a, b, c\}, \{a, d, e\}$ exactly once, it has the form $\{a, u, v\}$ with $u, v \in \operatorname{Int}(B_1)$. The same lemma gives disjoint b–u and d–v paths. Choose p', q', r, z as the four twins. The set $\operatorname{Int}(B) \cup \{b, c, p, q\}$ is connected because c has a neighbor in the interior of B and p, q join b to c. It lies in a component outside the new Ψ_3, with neighbors at a, d, e through B and at u through the b–u path. Since five attachments are excluded, its bridge has exactly four attachments and at least $|\operatorname{Int}(B)| + 4$ interior vertices, contrary to the choice of B. Thus $B_1 + ab + ad$ is nonplanar, and it has no K_5 minor. Let B_1' be an allowed minor of B_1 with $B_1' + ab + ad$ nonplanar. Contract the interior of B to w and keep only Ψ_3, B_1' and the edges at w. The resulting minor is $\Psi_3 \cup B_1' + w$, where $N(w) = \{a, c, d, e\}$. The graph $\Psi_3 + w$ contains $K_{3,3}$ with parts $\{a, d, e\}$ and $\{r, z, w\}$. Thus, in every torus embedding of $\Psi_3 + w$, each component of the complement of its closed regular neighborhood is a disk by Lemma 2.1. In every such embedding, a facial walk containing a, b, d has attachment sequence $abad$, up to cyclic shift and reversal [Supp. R7]. Lemma 2.2 therefore requires $B_1' + ab + ad$ to be planar, a contradiction. Hence this minor is nontoroidal. Taking $B_1' = B_1$, minimality gives $G = \Psi_3 \cup B_1 + w$, and Corollary 3.19, with attachments a, b, d and added edges ab, ad, gives 349 possibilities for B_1. These are the 349 forms of Proposition 6.13 again. Every resulting graph has a K_5 minor, a cut of at most two vertices, or a proper nontoroidal minor [Supp. E23]. Thus every other bridge with nonempty interior has planar triangle completion, and Lemma 3.2 makes its interior a single vertex. □

The next lemma excludes the additional vertices and gives the two nonplanar completions of B.

Lemma 6.15. *With d, e labeled as in Lemma 6.14, $G = (\Psi_3 \cup B) + bd + E$, where E is $\varnothing$ or $\{cd\}$. The following graphs are nonplanar.*

$$B + ac + E + de, \qquad B + ac + E + ad + ae. \tag{3}$$

Proof. By Lemma 6.14, $n_c = 2$. If $n_e \geq 4$, contract the interior of B to a vertex and the outside b–d path into d. Keeping four vertices with neighborhood $\{a, d, e\}$ and deleting extra edges gives the proper $\mathfrak{T}_2$ minor of [Supp. W16]. Thus $n_e = 2$ or 3. There is at most one vertex with each neighborhood $\{a, b, d\}, \{a, c, d\}$, since two would give a twin triple meeting $\{a, b, c\}$ twice. Such a vertex excludes bd or cd, respectively, by independence of its neighborhood. The outside path requires bd or a vertex with neighborhood $\{a, b, d\}$. There are six possible choices of additional edges and vertices, listed in the first column below, each with $n_e = 2$ or 3. A triple denotes one additional vertex with that neighborhood. For each of these twelve choices, G is toroidal exactly when adding one of the edge sets in the corresponding table entry to B gives a planar graph.

Additional edges or neighborhoods	$n_e = 2$	$n_e = 3$
bd, or $\{a, b, d\}$	$\{ac, de\}$ or $\{ac, ad, ae\}$	$\{ac, ad, de, ae\}$
bd, cd, or $\{a, b, d\}$ and cd	$\{ac, cd, de\}$ or $\{ac, cd, ad, ae\}$	$\{ac, cd, de, ae\}$
$\{a, c, d\}$ and bd, or $\{a, b, d\}, \{a, c, d\}$	$\{ac, cd, ad, ae\}$ or $\{ac, cd, ad, de\}$ or $\{ac, cd, de, ae\}$	$\{ac, cd, de, ae\}$

For each of these twelve choices, the attachment sequences from all rotations of $G - \mathrm{Int}(B)$ are recorded in [Supp. R8]. In genus one, join distinct consecutive attachments on the boundary of each face meeting all four. By Lemma 2.2, B can be drawn in that face exactly when adding these edges to B gives a planar graph. In genus zero, no boundary walk contains all four attachments. The only possible annulus meeting all four is $ac \mid de$, in which B can be drawn exactly when $B + ac + de$ is planar by Lemma 2.4. Keeping the inclusion-minimal edge sets for each choice gives the table.

If a vertex has neighborhood $\{a, b, d\}$, contract it into b and delete the new edge ab. This replaces it by bd without changing the tests in its row, giving a proper nontoroidal minor. Thus no such vertex remains, and bd is an edge. Suppose a vertex has neighborhood $\{a, c, d\}$. If $n_e = 3$, contract it into d and delete the new edge ad. This replaces it by cd, changing the last row to the second with the same single test. If $n_e = 2$, contract bd into d, leaving B unchanged. There are now three vertices with neighborhood $\{a, c, d\}$ and two with $\{a, d, e\}$. Lemma 5.3, with (a, b, c, d) replaced by (a, d, c, e), gives the first two tests in the last row, both nonplanar. In either case, the resulting proper minor is nontoroidal. Thus neither additional neighborhood occurs.

Suppose $n_e = 3$. The applicable row gives a single nonplanar completion $B + E'$. By Lemma 6.2 with $U = B \cup bd$, contracting bd into d gives the minor $B + ac + ad + ae + cd + de$ of G. Hence $B + E'$ has no K_5 minor. Replacing B by an allowed minor B' preserves the connected interior and all four attachments, so the resulting minor is nontoroidal whenever $B' + E'$ is nonplanar. Corollary 3.19, with attachments a, c, d, e and added edges E', gives 9,127 possibilities without cd and 8,967 with cd. Every resulting graph has a proper nontoroidal minor obtained by deleting or contracting one edge [Supp. E24], a contradiction. Thus $n_e = 2$, and the first two rows give the claimed form of G and its two nonplanar completions. □

We now reduce B while keeping both completions in (3) nonplanar.

Proposition 6.16. *If no choice of the four twins gives a bridge with five attachments or four attachments excluding their common vertex, but some choice gives a bridge with four attachments including it, then G is isomorphic to one of $\mathfrak{T}_{14}, \mathfrak{T}_{37}, \mathfrak{T}_{41}, \mathfrak{T}_{59}$.*

Proof. Put $H_1 = B + ac + ad + ae + cd + de$. By Lemma 6.2 with $U = B \cup bd$, contracting bd into d gives H_1 as a minor of G, so it has no K_5 minor. It contains $G[V(B)] = B + E$, and each component of $G - V(B)$ has its neighbors in one of the triangles acd, ade. As in the proof of Lemma 6.12, any cut of H_1 therefore also separates G, so H_1 is three-connected.

If $B + ac + E$ is nonplanar, choose a $K_{3,3}$ subdivision $H_2 \subseteq B + ac + E$, which lies in both completions in (3). Suppose instead that $B + ac + E$ is planar, and regard d, e, a as its attachments. These are independent, and its interior $\mathrm{Int}(B) \cup \{c\}$ is connected and adjacent to all three. Its triangle completion is H_1 if $E = \{cd\}$ and $H_1 - cd$ otherwise. The latter is two-connected, since $H_1 - v$ is two-connected for every vertex v and remains connected after deleting cd. Any two-cut of $H_1 - cd$ avoids and separates c, d, since adding cd reconnects the graph. The path cad forces it to contain a, and its other vertex cannot be e, since $B[\mathrm{Int}(B) \cup \{c, d\}]$ is connected. Thus every two-cut contains a and avoids d, e. Both $B + ac + E + de$ and $B + ac + E + ad + ae$ are nonplanar by (3), so Lemma 3.16, with $(x, y, z) = (d, e, a)$, gives a $K_{3,3}$ subdivision $H_2 \subseteq B + ac + E + de$ using de such that $H_2 - de + ad + ae$ is nonplanar. Set $E' = \{ac\} \cup E$ when $B + ac + E$ is nonplanar and $E' = \{ac, de\} \cup E$ when it is planar.

Apply Lemma 3.17 with attachments a, c, d, e and added edges E'. This gives an allowed minor B' with at most nine interior vertices and a subdivision $H_2' \subseteq B' + E'$. When $B + ac + E$ is nonplanar, H_2' lies in both reduced completions. When $B + ac + E$ is planar, the reduction keeps de and the position or absence of a, so Lemma 2.6 makes $H_2' - de + ad + ae$ nonplanar. This graph lies in the second completion, while H_2' lies in the first. Thus both remain nonplanar, and $G' = (\Psi_3 \cup B') + bd + E$ is a nontoroidal minor with at most 18 vertices. Minimality gives $G' = G$.

Construction 3.18, using E' and requiring de when $B + ac + E$ is planar, therefore includes every possibility. The two choices of E and the two cases for $B + ac + E$ give 66,384 candidates. Every resulting graph is toroidal, has a K_5 minor or a proper nontoroidal minor, or is isomorphic to one of thirteen graphs $\mathfrak{T}_i$ [Supp. E25]. Nine have twin triples meeting in zero or two vertices, leaving $\mathfrak{T}_{14}, \mathfrak{T}_{37}, \mathfrak{T}_{41}, \mathfrak{T}_{59}$. □

6.5 Every remaining bridge has three attachments

From now on, for every choice of the four twins, every Ψ_3-bridge with nonempty interior has exactly three attachments. Fix such a choice. We first restrict the possible neighborhoods and then show that every bridge with nonempty interior has a planar completion by two edges, allowing us to apply Proposition 3.11.

Lemma 6.17. *After relabeling within $\{b,c\}$ and $\{d,e\}$, the pairs xy with $x \in \{b,c\}$ and $y \in \{d,e\}$ joined by an edge or a component form one of the sets*

$$\{bd, ce\} \quad \text{or} \quad \{bd, cd, ce\}. \tag{4}$$

Also, $n_c + n_e \le 6$, and every bridge with nonempty interior has a three-connected, K_5-minor-free triangle completion.

Proof. If b is joined to neither d nor e by an edge or a component of $G - V(\Psi_3)$, then deleting a, c separates b from d, e. Thus three-connectivity requires each vertex of $\{b,c\}$ to be joined to a vertex of $\{d,e\}$, and conversely. Contract J_1, J_2 to the triangles abc, ade. Each component meeting both sets has only three attachments, so one of the sets contains just one of its attachments. Contract the component into that attachment. This gives an edge for every pair joined by the component. If all four pairs occurred, we would now have a K_5 on a, b, c, d, e. Since every vertex occurs in a pair, the pairs form either a matching of size two or a path of length three. After relabeling, these are the two possibilities in (4). In the first, the possible component neighborhoods are $\{a,b,c\}$, $\{a,d,e\}$, $\{a,b,d\}$ and $\{a,c,e\}$. The second also allows $\{a,c,d\}$, $\{b,c,d\}$ and $\{c,d,e\}$.

In either case, choose a b–d path and a c–e path in G, each consisting of a cross edge or passing through a single component of $G - V(\Psi_3)$. The paths are disjoint, since a component supplying both would have four attachments. They avoid a and all vertices counted by n_c, n_e, whose neighborhoods are $\{a,b,c\}$ or $\{a,d,e\}$. Contract the first path to identify b, d and the second to identify c, e. Each of the $n_c + n_e$ vertices is now adjacent to a and to both identified vertices. If $n_c + n_e \ge 7$, this gives a nontoroidal $K_{3,7}$ minor. It is proper since we identified distinct vertices of G, contrary to minimality. Hence $n_c + n_e \le 6$.

Let B be a bridge with nonempty interior. Its triangle completion is three-connected by Lemma 3.1. Apply Lemma 6.2 to add the triangles abc, ade while retaining all bridges. For attachments $\{a,b,c\}$ or $\{a,d,e\}$, the required triangle is already present. For $\{b,c,d\}$ or $\{c,d,e\}$, contract a into d or c, respectively. For $\{a,b,d\}$, follow bc, the chosen c–e path, and ed to join b to d. For $\{a,c,d\}$ or $\{a,c,e\}$, follow cb and the chosen b–d path, continuing along de if the attachment is e. In each case the chosen path has an end outside the attachment set of B, so it avoids the interior of B. Shortening the resulting path to one edge completes the triangle, since a is already adjacent to both ends. Thus the triangle completion of B is a minor of G and has no K_5 minor. □

We first consider bridges with attachments $\{b,c,d\}$ or $\{c,d,e\}$, and then a bridge with attachments $\{a,b,d\}$.

Lemma 6.18. *Every bridge with nonempty interior and attachments $\{b,c,d\}$ or $\{c,d,e\}$ has a planar completion by two edges.*

Proof. Exchanging b, e and c, d together if necessary, suppose B has attachments $A = \{b,c,d\}$ and no planar completion by two edges. Its triangle completion is therefore nonplanar, and by Lemma 6.17 it is a three-connected minor of G. Suppose $B + bc$ is planar. Lemma 3.8 gives a three-cut X of the triangle completion and two interior vertices with neighborhood X. This twin triple meets each of $\{a,b,c\}$ and $\{a,d,e\}$ exactly once. Since neither a nor e belongs to B, X contains d and exactly one of b, c. By part (vi) of the same lemma, adding an edge between two vertices of $A \cap X$ preserves planarity. Thus, if $b \in X$, then $B + bc + bd$ is planar, and if $c \in X$, then $B + bc + cd$ is planar. Both contradict our assumption. Hence $B + bc$ is nonplanar.

Consider a torus embedding of $(\Psi_3 \cup B) + ce$. If a closed regular neighborhood of $\Psi_3 + ce$ has genus one, the facial walks containing b, c, d have attachment sequences bcd or $bcdc$ [Supp. R9]. Both disk tests require $B + bc$ to be planar. If the neighborhood has genus zero, no boundary walk contains all three attachments, so B must lie in an annulus. The only possible boundary sequences are $bc \mid ad$ and $bc \mid de$ [Supp. R9]. By Lemma 2.4, the corresponding tests are $B + bc + ad$ and $B + bc + de$. Each is planar exactly when $B + bc$ is planar, since the extra vertex a or e has degree one. Conversely, take the plane embedding of $\Psi_3 + ce$ in [Supp. R9] with b, c on one facial walk and a, d on another. Remove an open disk from each face and join the two boundary circles by a cylinder. If $B + bc$ is planar, Lemma 2.4 places B in the resulting annulus $bc \mid ad$. Thus $(\Psi_3 \cup B) + ce$ is toroidal exactly when $B + bc$ is planar.

By Lemma 6.17, ce is an edge or is supplied by a component outside Ψ_3. Since e is not an attachment of B, we may shorten the resulting c–e path to ce and retain $(\Psi_3 \cup B) + ce$ as a minor of G. The same embedding test applies after replacing B by any allowed minor B', so this minor remains nontoroidal whenever $B' + bc$ is nonplanar. By minimality and Corollary 3.19, B is among the 563 forms in [Supp. E26] with added edge bc. Every resulting graph has a proper nontoroidal minor except one copy of $\mathfrak{T}_{41}$, for which $B + bd + cd$ is planar. This contradicts our assumption. □

Lemma 6.19. *Suppose a bridge B with attachments $\{a, b, d\}$ has no planar completion by two edges. Then every other bridge with nonempty interior has attachments $\{a, b, d\}$ or $\{a, c, e\}$, and every cross edge is bd or ce.*

Proof. First suppose another bridge has attachments $\{a, b, c\}$, $\{a, d, e\}$, $\{a, c, d\}$ or $\{c, d, e\}$. Contract its interior to a vertex v. In the first three cases, (4) gives the edge ce or a c–e path through a component outside Ψ_3. Neither B nor the bridge contracted to v has both c, e as attachments, so we may shorten this path to ce while keeping both bridges intact up to the contraction to v. Retain Ψ_3, B, the edges at v, and ce in the first three cases. In each case the retained graph outside $\mathrm{Int}(B)$ is nonplanar, so every face in a torus embedding of that graph is a disk. The facial walks containing a, b, d have sequence $abad$, with abd also possible when $N(v) = \{a, c, d\}$ [Supp. R10]. The test for $abad$ is $B + ab + ad$, and the test for abd is the triangle completion, which contains $B + ab + ad$. Since $B + ab + ad$ is nonplanar, Lemma 2.2 makes the retained minor nontoroidal. By minimality it is G. The same argument applies to any allowed minor B' for which $B' + ab + ad$ is nonplanar. Thus Corollary 3.19 gives the 349 forms with added edges ab, ad. Every resulting graph has a proper nontoroidal minor except two with $N(v) = \{c, d, e\}$. Both have $B + ad + bd$ planar [Supp. E27], contrary to hypothesis.

It remains to rule out a bridge on $\{b, c, d\}$ and the edge cd. If there is such a bridge, contract its interior to v. As above, we may retain ce, since this bridge has no attachment at e. If cd is an edge, retain cd and ce. The resulting minor is, respectively,

$$H_1 = (\Psi_3 \cup B) + ce + v \qquad \text{or} \qquad H_1 = (\Psi_3 \cup B) + ce + cd.$$

In either graph, r, z are adjacent to each of a, d, e, and c joins those same three vertices by cpa, by cvd or cd, and by ce. These paths give a $K_{3,3}$ subdivision in $H_1 - \mathrm{Int}(B)$, so every face in a torus embedding of this graph is a disk. For each choice of H_1, the facial sequences containing a, b, d are $abad$, $adbd$ and abd [Supp. R11]. The first two test $B + ab + ad$ and $B + ad + bd$, respectively. The last tests the triangle completion, which contains both. Thus Lemma 2.2 gives

$$H_1 \text{ is toroidal} \iff B + ab + ad \text{ is planar or } B + ad + bd \text{ is planar.} \tag{5}$$

Both completions are nonplanar, so H_1 is nontoroidal and minimality gives $G = H_1$.

Suppose $B + ad$ is nonplanar. If an allowed minor B' keeps $B' + ad$ nonplanar, then both completions in (5) remain nonplanar. Replacing B by B' therefore keeps H_1 nontoroidal, and Corollary 3.19 gives the 563 forms with added edge ad. Every resulting graph has a K_5 minor or a proper nontoroidal minor except four in the cd case, $\mathfrak{T}_{52}, \mathfrak{T}_{57}, \mathfrak{T}_{58}, \mathfrak{T}_{59}$. In the first three, some two twin triples meet in zero or two vertices. In the last, some choice of the four twins gives a bridge with four attachments [Supp. E28]. Each contradicts our assumptions, so $B + ad$ is planar.

By Lemma 6.17, the triangle completion of B is three-connected and has no K_5 minor. By (5), no proper allowed minor of B keeps both $B + ab + ad$ and $B + ad + bd$ nonplanar. We may therefore apply Lemma 3.14 with $(x, y, z) = (a, d, b)$. Since $B + ab + bd$ is also nonplanar, only the two four-cycle forms remain. In both, the interior of B is the cycle $up'u'q'u$ and b is adjacent to u, u'. If d is adjacent to p', q' and a only to u', then p', q' are twins with neighborhood $\{d, u, u'\}$, disjoint from the twin triple $\{a, b, c\}$.

In the other form, p', q' are twins with neighborhood $\{a, u, u'\}$ and d is adjacent only to u' in B. Choose p', q' and r, z as the four twins. The old vertices b, c, p, q lie in one component outside the new Ψ_3. This component meets u, u' through b, a through p, and e through ce. It also meets d through cd or cvd. Its bridge therefore has five attachments, contrary to the assumption that every choice of the four twins gives only three-attachment bridges with nonempty interior. This rules out both forms. By Lemma 6.17, the only remaining attachment triples are $\{a, b, d\}, \{a, c, e\}$, and the only remaining cross edges are bd, ce. □

The next lemma gives the planar completions needed for Proposition 3.11.

Lemma 6.20. *The attachments of every bridge B with nonempty interior can be labeled x, y, t so that $B + xy + xt$ is planar.*

Proof. Suppose a bridge B has no planar completion by two edges. First suppose its attachments are $\{a, b, c\}$, $\{a, d, e\}$ or $\{a, c, d\}$. By (4), an edge or a component joins b to d, and likewise c to e. Neither path can pass through the interior of B, since B contains only one end of each. Their interiors are disjoint, since a component supplying both paths would have four attachments. Contract each path to its end in B and retain only B and the edges at p, q, r, z. This identifies b with d and c with e, so all four twins now have the attachment triple of B as their neighborhood. The minor is proper, hence toroidal, and Lemma 4.18 gives a planar completion of B by two edges, a contradiction.

Lemma 6.18 excludes $\{b,c,d\}$ and $\{c,d,e\}$. The two remaining triples $\{a,b,d\}$ and $\{a,c,e\}$ are interchanged by exchanging b,e and c,d, so suppose B has attachments $\{a,b,d\}$. By Lemma 6.19, every bridge with nonempty interior has one of these two triples, and only the cross edges bd, ce can occur. Let B_{bd} consist of the bridges on $\{a,b,d\}$ together with bd if present, and define B_{ce} similarly. Suppose $B_{ce}+ac+ae$ is nonplanar. In $B+ab+ad$ replace ab, ad by apb, ard, and in $B_{ce}+ac+ae$ replace ac, ae by aqc, aze. This gives two nonplanar subgraphs of G meeting only at a. Their union has genus at least two by block additivity, yet it is a proper subgraph of G since it omits pc. This contradicts minimality, so $B_{ce}+ac+ae$ is planar.

By (4), $B_{ce}-a$ contains a c–e path. Contract it to a vertex f and retain only B_{bd} and the edges at p,q,r,z. This is a proper minor of G, so fix a torus embedding of it. The edges at the four twins now form a copy H of Ψ_2 with twin triples $\{a,b,f\}$ and $\{a,d,f\}$. Since B joins b,d, a closed regular neighborhood of H of genus one would give a planar two-edge completion of B by Lemma 5.29. Thus the neighborhood has genus zero.

The planar rotation of H has six facial walks by Euler's formula. Their lengths sum to 24, and each has even length at least four, so all six are quadrilaterals. The two walks through b pass through p,q, one closing through a and the other through f. Similarly, the two through d pass through r,z, one closing through a and the other through f. The six attachment sequences are therefore ab, bf, ad, df, af, af. Every bridge in B_{bd} meets both b,d, as does the edge bd if present. Since no boundary walk contains both, each must lie in a region with at least two boundary components. By Lemma 2.1, this region is the unique annulus. One boundary contains b and the other contains d, so the two af walks bound disks whose interiors are disjoint from B_{bd}.

At f, the bf walk makes fp, fq consecutive and the df walk makes fr, fz consecutive. The two af disks lie between these pairs. Split f into c,e joined by an edge, keeping p,q at c and r,z at e. The two disks now share ce, one with attachment sequence ace and the other with sequence aec. Deleting ce joins them into a disk with boundary sequence $caea$. In a plane drawing of $B_{ce}+ac+ae$, cut the sphere along cae and discard the two added edges. This gives a drawing of B_{ce} in a disk with the same boundary sequence $caea$, with a appearing once on each side of the cut. Insert this drawing into the disk just obtained. Together with Ψ_3 and B_{bd}, it gives a torus embedding of G, a contradiction. □

6.6 Bounding the size and number of the remaining bridges

By Lemma 6.20, label the attachments of each bridge B with nonempty interior as x,y,t so that $B+xy+xt$ is planar. Then $B-x$ is a chain of edges and four-cycles by Proposition 3.11. We first consider $x \neq a$.

Lemma 6.21. *If $x \neq a$, then B is the three-edge star or $B-x$ has one of the following six forms, with end lengths as in Lemma 5.32.*

(i) *One four-cycle with end lengths $(0,1),(0,2),(1,0)$ or $(2,0)$.*

(ii) *Two four-cycles with end lengths $(0,0)$, joined by a path of length 1 or 2.*

Each attachment on a cycle belongs to $\{d,e\}$ if $x \in \{b,c\}$, and to $\{b,c\}$ if $x \in \{d,e\}$. In each case, B has at most seven interior vertices.

Proof. If $B-x$ has no cycle, Proposition 3.11(v) gives the three-edge star. Otherwise, temporarily interchange the two twin triples if $x \in \{d,e\}$, then interchange b,c if necessary so that $x=b$. Each cycle gives twins with neighborhood $\{b,s,u\}$. Here s is the vertex shared with the preceding block, or the first end attachment if the cycle is the first block, and u is defined in the same way using the following block. They cannot both be end attachments, since the cycle would then be the whole chain and the interior of B would be disconnected. Thus this twin triple differs from $\{a,b,c\}$ and $\{a,d,e\}$. Since it meets each in exactly one vertex, it contains neither a nor c and contains exactly one of d,e. Every cycle therefore contains exactly one end attachment, lying in $\{d,e\}$, so there are at most two cycles. With one cycle, one end length is zero and the other is one or two by Proposition 3.11(iii), giving (i). With two cycles, both end lengths are zero. The cycles cannot share a vertex, since their twin triples would then intersect in both that vertex and b. Their joining path therefore has length one or two, giving (ii). The forms in (i) have at most four interior vertices, and those in (ii) have six or seven. □

Lemma 6.22. *At most two bridges have nonplanar triangle completion, and none has attachments $\{a,b,c\}$ or $\{a,d,e\}$. Among these bridges, the choices $x=a$ and $x \neq a$ cannot occur together. If $x=a$ occurs, the set in (4) must be $\{bd,ce\}$, and each such bridge with attachments a,y,t excludes the edge yt and every other component with both y,t as neighbors. If $x \neq a$ occurs, these bridges contain at most one four-cycle in total when that set is $\{bd,ce\}$, and at most two when it is $\{bd,cd,ce\}$.*

Proof. Suppose B has nonplanar triangle completion and $x=a$. By Proposition 3.11, the chain $B-a$ contains a four-cycle. Write any such cycle as $up'vq'u$, where p',q' have neighborhood $\{a,u,v\}$. If both u,v were end

attachments, this cycle would be the whole chain, leaving p', q' as the two components of the interior of B. If exactly one were an end attachment, then $\{a, u, v\}$ would meet the original twin triple containing that attachment in exactly two vertices. Thus every cycle avoids both end attachments y, t, so y, t have degree one in B. Neither can serve as x, since the same proposition would then give a four-cycle with two twins adjacent to that vertex.

Fix one such cycle and label u, v so that the chain gives disjoint y–u and t–v paths. Suppose B has attachments $\{a, b, c\}$. Choose p', q' and r, z as the four twins. The component outside the new Ψ_3 containing b, c, p, q has a neighbor at a through p and at u, v along the two paths. By (4), b also joins one of d, e by an edge or a component outside B. This gives a fourth attachment, a contradiction. Interchanging the original twin pairs likewise excludes $\{a, d, e\}$.

The two end attachments therefore lie one in $\{b, c\}$ and one in $\{d, e\}$. Temporarily label them b, d, with the chain paths running from b to u and from d to v. Again choose p', q' and r, z. The component containing b, c, p, q has neighbors at a through p and at u along the b–u path. It also has a neighbor at e, since e joins b or c by an edge or a component and e is not an attachment of B. Any edge or component outside B joining d to b or c would now give a fourth attachment at d. Choose instead p', q' and p, q. The component outside this copy of Ψ_3 containing d, e, r, z has neighbors at a through r, at v along the d–v path, and at c since c joins d or e outside B. Thus no edge or component outside B joins b to d or e. Together these exclusions leave only the pairs bd, ce, with B the only bridge joining b, d and the edge bd absent. Each bridge with $x = a$ therefore accounts for one of the two pairs in the first alternative of (4), and no two such bridges account for the same pair. Hence there are at most two.

Now suppose $x \neq a$. By Lemma 6.21, each four-cycle in the chain gives a twin triple consisting of one attachment from $\{b, c\}$, one from $\{d, e\}$, and one interior vertex. In particular, this excludes the attachment sets $\{a, b, c\}$ and $\{a, d, e\}$. Different cycles give different interior vertices, whether they lie in the same chain or in different bridges. Since their twin triples meet exactly once, their attachment pairs must meet in exactly one vertex. Thus no pair occurs twice, and bd and ce cannot both occur. This gives at most one cycle when the available pairs are $\{bd, ce\}$ and at most two when they are $\{bd, cd, ce\}$. Every bridge with nonplanar triangle completion has a cycle in its chain, so the same bounds hold for the number of bridges. Finally, a cycle triple from a bridge with $x = a$ consists of a and two interior vertices of that bridge. A cycle triple from another bridge with $x \neq a$ would contain none of these three vertices, but any two distinct twin triples meet in exactly one vertex. Thus the two choices cannot occur together. □

It remains to determine the bridges with $x = a$.

Lemma 6.23. *Let B be a bridge with attachments a, y, t. If its triangle completion is nonplanar and $B + ay + at$ is planar, then $B = Q(a; y, t)$.*

Proof. By Lemma 6.22, the pairs in (4) are bd, ce. Interchanging b, c and d, e simultaneously if needed, suppose B has attachments $\{a, b, d\}$. Put $H_1 = G - \mathrm{Int}(B)$. No edge or bridge of H_1 joins b, d. Suppose H_1 is planar. If an additional bridge had attachments $\{a, b, c\}$, contracting its interior to a vertex would give a $K_{3,3}$ with p, q and a, b, c. The same argument with r, z excludes $\{a, d, e\}$. Thus every bridge remaining outside Ψ_3 in H_1 has attachments $\{a, c, e\}$ or is the edge ce. Let B_{ce} be their union on a, c, e. Shortening apc to ac and are to ae gives $B_{ce} + ac + ae$ as a planar minor of H_1. By hypothesis, $B + ab + ad$ is also planar. Draw these two graphs in disks meeting only at a, with the four added edges in cyclic order ab, ad, ac, ae. Since ab, ac alternate with ad, ae, Corollary 6.7 places $B \cup B_{ce}$ in the disk face with sequence $abadacae$ of a torus embedding of Ψ_3. This embeds G in the torus, a contradiction. Thus H_1 is nonplanar.

By Proposition 3.11, the chain $B - a$ contains a four-cycle, and by the proof of Lemma 6.22, every such cycle avoids b, d. Two consecutive cycles cannot share a vertex, since their twin triples would then meet at both a and that vertex. Thus each end path and each path between consecutive cycles has positive length, and each has length at most two since no three consecutive blocks are edges. If there is only one cycle and both end paths have length one, then $B = Q(a; b, d)$. Otherwise, shorten the path between two consecutive cycles, or, if there is only one cycle, shorten an end path of length two, as in Lemma 5.31. The resulting bridge B' still has connected interior and attachments a, b, d, and $H_1 \cup B'$ is a proper minor of G.

Take a torus embedding of $H_1 \cup B'$. Since H_1 is connected and nonplanar, its regular neighborhood has genus one, so every face of H_1 is a disk by Lemma 2.1. The connected interior of B' lies in one such face F. Represent F by a disk with a separate copy of each attachment at each boundary appearance used by B'. The boundary cycle contains all these copies and avoids the interior of B', and no edge outside B' meets its interior. Taking this cycle as H in Lemma 5.31 restores B in the disk with the same attachment copies. Putting the disk back in F gives a torus embedding of $H_1 \cup B = G$, a contradiction. Hence $B = Q(a; b, d)$. □

6.7 Combining the remaining bridges

The preceding lemmas give the following construction.

Construction 6.24. Start with $A = \{a, b, c, d, e\}$ and choose $M = \{bd, ce\}$ or $M = \{bd, cd, ce\}$ as in (4). Besides the two twin triples $\{a, b, c\}$ and $\{a, d, e\}$, we may use $\{a, b, d\}$ and $\{a, c, e\}$ as attachment triples. When $cd \in M$, we may also use $\{a, c, d\}$, $\{b, c, d\}$ and $\{c, d, e\}$. Give different bridges disjoint interiors.

(1) Choose zero, one or two bridges with nonplanar triangle completion. For each bridge B, choose one of the additional triples and consider every labeling x, y, t of its attachments. By Lemma 6.22, if one chosen bridge has $x = a$, then all do and $M = \{bd, ce\}$. In this case use $B = Q(a; y, t)$ by Lemma 6.23.

Otherwise, for each chosen bridge use one of the six forms for $B - x$ in Lemma 6.21, with end attachments y, t. The end attachment on each four-cycle lies in $\{d, e\}$ if $x \in \{b, c\}$, and in $\{b, c\}$ if $x \in \{d, e\}$. Join x to every vertex of degree two in $B - x$ other than y, t, as in Proposition 3.11. Allow at most one four-cycle in total among these chains if $M = \{bd, ce\}$, and at most two if $M = \{bd, cd, ce\}$. In either case, any two distinct twin triples among those in the chosen bridges and the two original twin triples must meet in exactly one vertex.

(2) For each additional triple, choose whether to add a single vertex with that neighborhood. Choose which edges of M to add. If $Q(a; y, t)$ was chosen in step (1), omit the edge yt and require that no other bridge have both y, t as attachments. Keep the choice precisely when, for each $u \in \{b, c\}$ and $v \in \{d, e\}$, we have $uv \in M$ if and only if uv is an edge or u, v have neighbors in the same component outside A.

(3) Add n_c vertices with neighborhood $\{a, b, c\}$ and n_e vertices with neighborhood $\{a, d, e\}$, taking

$$(n_c, n_e) \in \{(2, 2), (2, 3), (2, 4), (3, 2), (4, 2)\}.$$

These counts include the two chosen twin pairs defining Ψ_3.

Treat a, b, c, d, e as labeled vertices and keep one representative of each isomorphism class.

Proposition 6.25. *If, for every choice of the four twins, every Ψ_3-bridge with nonempty interior has exactly three attachments, then G is isomorphic to one of $\mathfrak{T}_7, \mathfrak{T}_{33}, \mathfrak{T}_{35}, \mathfrak{T}_{49}, \mathfrak{T}_{70}$.*

Proof. The preceding lemmas give all bridges with nonplanar triangle completion in step (1). By Lemma 3.2, every other bridge with nonempty interior is a three-edge star. Two such stars on the same additional triple would give twins whose common neighborhood meets $\{a, b, c\}$ or $\{a, d, e\}$ in two vertices, a contradiction. Thus step (2) includes all stars on the additional triples and all possible cross edges. For the two original triples, Lemmas 6.5 and 6.17 give $\min(n_c, n_e) = 2$ and $n_c + n_e \leq 6$, which give the five choices in step (3). Hence Construction 6.24 includes G.

For $n_c = n_e = 2$, this gives 31,063 candidates. By Lemma 3.4(a), we exclude any candidate having a degree-three vertex with two adjacent neighbors. We also exclude any candidate for which some choice of the four twins gives a bridge with four or five attachments. Both exclusions still apply when n_c or n_e increases, since each new vertex is adjacent only to vertices among a, b, c, d, e. In the first exclusion, the recorded degree-three vertex lies outside these five, so it keeps the same three neighbors. In the second, the four twins used in the check also lie outside these five, so they still define the same copy of Ψ_3. The old component lies in a component outside this copy that still has all four or five original attachments.

This leaves 193 candidates for $n_c = n_e = 2$, giving 965 cases over the five choices of (n_c, n_e). Of these, 172 are toroidal and 782 have a proper nontoroidal minor. The remaining 11 are copies of the five graphs in the statement [Supp. E29]. □

Proof of Theorem 6.1. Propositions 6.8, 6.13 and 6.16 treat every graph for which some choice of the four twins gives a bridge with five or four attachments. Otherwise, for every choice, every bridge with nonempty interior has three attachments, so Proposition 6.25 applies. The converse follows from Proposition 7.1. □

7 Completion of the classification

We first show that the listed graphs have the required properties. The computational details are in Appendix A.

Proposition 7.1. *The graphs $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$ are pairwise nonisomorphic. Each has no K_5 minor, has genus two, and has every proper minor toroidal. For each of Theorems 4.1, 5.1 and 6.1, every graph in its list is three-connected and satisfies its stated conditions on twin triples.*

Proof. The checks in [Supp. O2] and Appendix A establish the stated genus, absence of a K_5 minor, nonisomorphism, connectivity and twin triples. They also give torus embeddings of every $G - e$ and G/e for each listed graph G. Since G has no isolated vertices, every proper minor is a minor of one of these graphs and is therefore toroidal. □

Proof of Theorem 1.1. Proposition 7.1 proves membership. Conversely, let G be a toroidal minor obstruction with no K_5 minor. If G is not two-connected, block additivity and minimality give two nonplanar blocks. Each contains a $K_{3,3}$ minor, since G has no K_5 minor. Their union therefore has a nontoroidal minor consisting of two copies of $K_{3,3}$, disjoint or sharing one vertex. Minimality makes G this minor, giving $\mathfrak{T}_{25}$ or $\mathfrak{T}_8$. The one-vertex union is unique because $K_{3,3}$ is vertex-transitive. Mohar and Škoda classified the 68 toroidal obstructions of connectivity two [18, Theorem 8.10 and Corollary 8.11]. Exactly 18 have no K_5 minor, and the comparison in [Supp. O3] identifies them with the listed graphs of connectivity two. If G is three-connected, either $G \cong K_{3,7} = \mathfrak{T}_1$ or Theorem 3.6 gives two distinct twin triples. If some two are disjoint, Theorem 4.1 applies. Otherwise, if some two share two vertices, Theorem 5.1 applies. In the remaining case every two distinct twin triples share exactly one vertex, so Theorem 6.1 applies. Thus G belongs to the list. □

8 Topological obstructions

A *topological obstruction* for the torus is a nontoroidal graph of minimum degree at least three that becomes toroidal when any edge is deleted. Every minor obstruction is one, since deleting a vertex of degree at most one or suppressing a degree-two vertex preserves genus.

A *split* replaces a vertex by two adjacent vertices and divides its neighbors between them, with at least two on each side. Every topological obstruction can be obtained from a minor obstruction by splits [3, Corollary 2.1.2].

Each intermediate graph H still has minimum degree at least three and contains the starting obstruction as a minor. Undoing the later splits by contraction gives H from the final graph G. For any edge e of H, the same contractions give $H - e$ from G with the corresponding edge deleted, so $H - e$ is toroidal. Thus H is also a topological obstruction. Since H is a minor of G, it has no K_5 minor whenever G has none.

Theorem 8.1. *There are exactly* 381 *topological obstructions for the torus with no K_5 minor, given in* [Supp. TO1]. *Their orders are as follows.*

n	10	11	12	13	14	15	16	17
number	1	9	43	111	151	58	7	1

Proof. Start with $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$ and split each vertex in every possible way. If the resulting graph has no K_5 minor and every edge deletion is toroidal, add it to the list unless an isomorphic graph is already there. Do the same for each new graph.

By Theorem 1.1 and the preceding paragraph, every topological obstruction with no K_5 minor can be reached from a starting graph by splits, and every graph along the way passes both tests. Each split adds one vertex and one edge, so $2|E| - 3|V|$ decreases by one. This quantity is at most 12 for the starting graphs and stays nonnegative since the minimum degree is at least three. Thus no sequence has more than 12 splits.

The splits give 7,006 graphs up to isomorphism [Supp. TO2]. Of these, 4,884 have a K_5 minor. Among those with no K_5 minor, 1,815 have an edge deletion containing one of the starting graphs as a minor. This leaves 307 topological obstructions. Each contains the graph it was split from as a nontoroidal proper minor, so none is among the 74 starting graphs. The total is $74 + 307 = 381$. □

9 Further remarks

Finding all the toroidal obstructions and proving that the list is complete seems difficult. Theorem 1.1 and the result of Gagarin, Myrvold and Chambers [8] settle the cases where one of the two forbidden minors for planarity is absent. By block additivity and the classification of the obstructions of connectivity two [18], any toroidal obstruction still to be found must also be three-connected.

One class to consider next is the toroidal obstructions with no K_5 subdivision. Those with no K_5 minor are already on our list. For those that do contain a K_5 minor, we can no longer use Halin's theorem to find the three-cuts that gave us the twin triples.

Let S be a closed connected surface other than the sphere. The proof of Theorem 3.6 shows that every three-connected minor obstruction for S with no K_5 minor is either $K_{3,k}$ for some k or has two distinct twin triples. The Wagner graph V_8 embeds in both the torus and the projective plane, so it embeds in S by adding handles

or crosscaps away from the graph. Even a noncellular embedding suffices to rule out V_8 as an obstruction when applying Halin's theorem. The disk replacement in Lemma 3.2 works on any surface. The neighbors of a degree-three vertex v are still independent, since if two were adjacent, we could embed the graph with their edge deleted and then draw it back alongside the path through v.

Acknowledgements

I would like to thank Dr. François Clément for advising me on the early work of this problem and for his helpful discussions, recommendations and feedback. I also thank Alexander Metzger for useful discussions and ideas, as well as his assistance with aspects of the computational work.

Declaration of AI Use

The author used OpenAI's ChatGPT and Codex (Astra, Sol and earlier GPT models) and Anthropic's Claude (Opus 4.8 and later Opus models, and Fable) to write scripts, help analyze their outputs, organize the repository, and review related literature. Some exploratory computations helped the author develop ideas for definitions, conjectures and statements to prove. AI also sometimes suggested approaches to arguments and generated TikZ code from the author's instructions and, in some cases, the author's hand-drawn sketches. The author decided which suggestions to use, and developed all the final statements and proofs. The author takes full responsibility for the paper, including its arguments, computations, references and conclusions.

Data Availability Statement

The supplemental data cited in this paper, and the genus computations and programs described in Appendix A, are available at `https://github.com/austinulrigg/toroidal-obstructions-no-k5`.

Conflict of Interest

The author declares no conflicts of interest.

A Computational details

The repository contains the supplemental data in the folder `supplemental_data`. Each item is named by its label in the text. For example, [Supp. R4] is the file `rotations/R4.json`.

The folder `no_K5_obstructions` contains O1–O4. O1 lists $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$. Table 2 gives them in graph6 format. The first character of a string gives the number of vertices, from `I` for ten to `M` for fourteen. For each graph G in O1, O2 gives a genus-two embedding of G, torus embeddings of $G - e$ and G/e for every edge e, and a certificate that G has no K_5 minor. It also records the connectivity and twin triples of G. The 74 graphs have distinct canonical forms, computed with nauty [15], and hence are pairwise nonisomorphic.

For comparison with Myrvold and Woodcock [23], we use the 17,535 graphs in `all_minor_obs.txt.gz` on their torus obstructions page, revised August 16, 2017.[2] O4 includes this collection, an isomorphism from each of $\mathfrak{T}_1, \ldots, \mathfrak{T}_{73}$ to a graph in it, and the canonical form of $\mathfrak{T}_{74}$, which does not occur in the collection. O3 lists the graphs of connectivity two in the collection. These are 68 pairwise nonisomorphic toroidal obstructions [23]. Since there are only 68 such obstructions [18], the list is complete. For 50 of them, O3 gives branch sets for a K_5 minor. The other 18 are isomorphic to the graphs of connectivity two in O1, which have no K_5 minor by O2.

Each table in R1–R12 lists every rotation system of each of its graphs, the genus of that rotation, and the attachments met by each facial walk in boundary order. For the enumerations E1–E29 in `enumerations`, the proof in the text shows that every graph in the case under consideration belongs to the stated finite family. The program generates every member of that family. Each item gives the construction and parameters used by the program, the candidates it produced, a witness for the outcome of each candidate, and the counts quoted in the text.

We use the same kinds of witnesses here and in W1–W16 in `witnesses`. For an embedding, we give the rotation system, trace its facial walks and compute its genus by Euler's formula. An isomorphism to some $\mathfrak{T}_i$ or to another named graph or bridge is given by a vertex bijection preserving adjacency in both directions. A minor model of $\mathfrak{T}_i$

[2] `https://webhome.cs.uvic.ca/~wendym/torus/torus_obstructions.html`

i	graph6	i	graph6	i	graph6	i	graph6	i	graph6
1	I???F~}~_	16	K????Bort}Nw	31	L?d_??B?v_QeYo	46	L@G[C@?OE?C~E}	61	L???EMeU@o?r}?
2	J?ACC@x^r}?	17	KC`A@?_w?~b}	32	L???WW?{DbMSrG	47	L@GcC@?OE?B~F}	62	Ms???????R_u^w^s?
3	J???wz_{FW?	18	KW????^Huozw	33	L???OCA{FWFktW	48	LB?CC?KwB@@xFu	63	MsG___K?q?G?BF?z?
4	JWr?__J@p`_	19	K?@Hc?B@Vp~E	34	L??`?_OqF_@|rw	49	M?@HcA??_G^?Q]K]?	64	Mr???????R_ufsZw?
5	J]o???fDo]?	20	KAO`C?X@s{[E	35	L????AL[cmLW~?	50	M@G[CA?OA?G?H~E~?	65	Mr????KB?E?WcNZo?
6	JBW?CMEw?{_	21	KC`A@?X@tLNA	36	L@G??B?@v?B|vo	51	M?d_GO???B_U{B{{?	66	ME?@@?_CWB@prArc?
7	J??EE?|Nvw?	22	KSP???rBu@fx	37	L?a??@bB?eno{S	52	M?o????AWNEo|C{w?	67	M??O`?_?]O]?Bwrc?
8	J??FF?[FFw?	23	KSP???rBuEF`	38	L?r@_??@WVEK{C	53	M?d``a?_A?G?BN?~?	68	M????cQo@W]?r_J[?
9	K????AN[vw^w	24	L?o???r@oO}p|Q	39	LI?GSA?o@o_|L{	54	LsP@@?OC?]@e?{	69	M?@b?oW_C??f?u}??
10	K????Bwfr]]w	25	Ks\o?CA?_B_M	40	L??E@_o?gYxa~?	55	L????Fas`YBg~?	70	MAP@@?__C?@fCufw?
11	K??@@B_v?~Zw	26	L???@KY?^GzA~_	41	L?KE?WC@F_EUo[	56	LAO`?A@_V?BxFk	71	MIK??KE_C?O?Q^Lf?
12	K???GGB}Fx^W	27	L@J???B?v?BxuK	42	MCaAA@?G@?C?F~F~?	57	L??@e?K?W[]B~C	72	M????B?Gow]?Bx~o?
13	K???GGC}Fv^[	28	L?AR?OGAF_Q]X[	43	L]o__O???J_u?{	58	L@G?E?E@opZAxa	73	M????B?Gow]?roNw?
14	K?CU@?BBVX]E	29	L???B?RB^c_f_	44	L??EE?cE?Ff`~?	59	L@G?E?E@op[Bv_	74	M??????gYk^?fwZw?
15	K?K??BoppmXw	30	L???GGww?wwX}o	45	LI?CC?K?xwUARp	60	L???C`_Bf_@|~_		

Table 2: The graphs $\mathfrak{T}_1, \ldots, \mathfrak{T}_{74}$ in graph6 format.

gives one nonempty connected vertex set for each vertex of $\mathfrak{T}_i$. These sets are disjoint, with an edge between two sets whenever the corresponding vertices are adjacent. For a K_5 minor, this gives five sets with an edge between every two. For a cut, we list at most two vertices whose deletion disconnects the graph. A subdivision of K_5 or $K_{3,3}$ certifies nonplanarity. To certify that a graph has no K_5 minor, we exhibit it as a subgraph of a graph obtained by gluing planar graphs and copies of V_8 along cliques of order at most three [28]. The certificate includes the pieces and a plane embedding of each planar piece.

When a candidate violates a hypothesis of the case, we record the reason, for example two distinct twin triples meeting in two vertices. When it has a proper nontoroidal minor, we give the edge to delete or contract and show why the resulting graph is nontoroidal. Either it has a minor model of some $\mathfrak{T}_i$, or it contains a graph of computed genus at least two, or it fails the embedding tests for its case. Each failed test is witnessed by a K_5 or $K_{3,3}$ subdivision or by a bridge that fits in none of the available faces.

In E3, E4 and E12, a graph is determined by the number of copies of each allowed bridge form and edge. If one combination uses at most as many copies of each as another, its graph is a subgraph of the other. To check that the recorded graphs cover all allowed combinations, we use a propositional formula whose variables encode these numbers. The clauses require that the combination be allowed, that no recorded toroidal graph contain it, and that it contain no recorded nontoroidal graph. For each of the three enumerations, `drat-trim` [29] checks a DRAT proof that this formula is unsatisfiable. Thus every allowed combination is a subgraph of a recorded toroidal graph or contains a recorded nontoroidal graph. The three items record 2,310, 315,099 and 9,878 graphs, respectively.

In `no_K5_topological_obstructions`, TO1 lists the 381 graphs of Theorem 8.1. TO2 records all 7,006 graphs produced by the search in its proof and the graph from which each was split. For each graph kept, TO2 gives a certificate that it has no K_5 minor and a torus embedding of each of its edge deletions. For each graph rejected, it gives either branch sets for a K_5 minor or an edge whose deletion contains some $\mathfrak{T}_i$ as a minor, together with the minor model.

N1 in `new_obstructions` gives the thirteen further obstructions mentioned in the introduction. For each graph G, it records its connectivity, branch sets for a K_5 minor, a genus-two embedding, and torus embeddings of $G - e$ and G/e for every edge e. None is isomorphic to a graph in the Myrvold–Woodcock collection.

We use exact genus computations with PAGE [16] and `multi_genus` [4] for the lower bounds on genus. The folder `genus` gives the results and running times. The repository also contains the programs for the rotation tables, the enumerations and the topological search.

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