Linearly Independent *Graphs?*

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Let's pretend for a moment that you have always desired to combine what are, naturally, your favorite two subjects, linear algebra and graph theory (as they are everyone's). Now, let's unnecessarily impose the condition that you also *hate eigenvectors* and hence have been dismayed by the seemingly impossibly vast disconnect between linear algebra and graph theory outside of spectral graph theory. If you are this extremely specific reader, then this post is exactly for you. Enjoy.

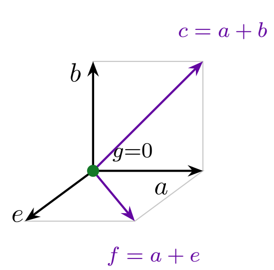
Linear Algebra

Consider the following vectors,

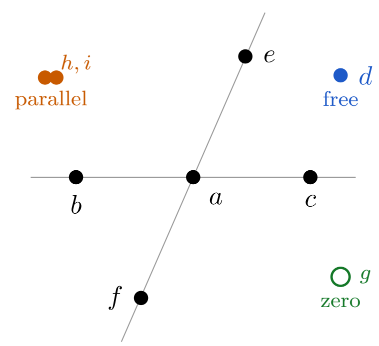
$$\vec{a} = (1,0,0,0,0), \quad \vec{b} = (0,1,0,0,0), \quad \vec{c} = (1,1,0,0,0), \quad \vec{d} = (0,0,0,1,0),$$
$$\vec{e} = (0,0,1,0,0), \quad \vec{f} = (1,0,1,0,0), \quad \vec{g} = (0,0,0,0,0), \quad \vec{h} = (0,0,0,0,1),$$
$$\vec{i} = (0,0,0,0,2).$$

One can easily notice that $\vec{a}, \vec{b}, \vec{c}, \vec{e}$, and $\vec{f}$ all have their last two coordinates equal to zero.

Because of this, we can view them as lying in $\mathbb{R}^3$ via their canonical projection onto it, as seen in Figure **??**(a). Representing these 5 vectors as being co-planar, we can also obtain the useful diagram (b) of all the vectors together.



(a) a, b, c, e, f in $\mathbb{R}^3$



(b) all nine vectors

Figure 1: Two views of the same nine vectors.

Suppose you are tasked with collecting all the independent sets of vectors from our collection. It is not hard to see that the points $\vec{a}, \vec{b}, \vec{c}$ are collinear, $\vec{a}, \vec{e}, \vec{f}$ are collinear as well, and the four points $\vec{b}, \vec{c}, \vec{e}, \vec{f}$, no three of them collinear, are dependent as a group of four. Additionally, the vectors $\vec{h}$ and $\vec{i}$ are parallel, $\vec{g} = 0$ is zero, and $\vec{d}$ is independent of everything. Therefore, we can list the minimal dependent sets,

$$\{\vec{g}\}, \quad \{\vec{h}, \vec{i}\}, \quad \{\vec{a}, \vec{b}, \vec{c}\}, \quad \{\vec{a}, \vec{e}, \vec{f}\}, \quad \{\vec{b}, \vec{c}, \vec{e}, \vec{f}\}.$$

Since a subset of vectors from our collection is linearly independent exactly when it contains none of these minimal dependent sets, we can be satisfied with our classification of the independent sets of our collection of vectors.

Graph Theory

Now, consider the following graph, G. Suppose that you are now tasked with collecting all of the sets of edges that form no cycles in G.

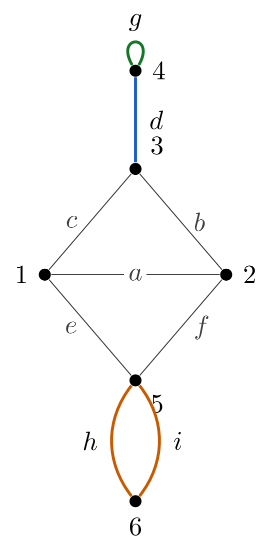


Figure 2: The graph G.

A clever observation is noting that by finding the edge sets which form the minimal cycles in G, we can determine the sets of edges which form no cycles, as they are exactly the edge sets containing none of the minimal cycles. That is,

$$\{g\}, \{h, i\}, \{a, b, c\}, \{a, e, f\}, \{b, c, e, f\}$$

So, while originally the questions could hardly look less alike, we have seen they have *exactly* the same answer: the same subsets are independent. Clearly, the example was artificially created to achieve such a result. However, it illustrates the point that there is an underlying structure to the concept of *independence* that is (no pun intended) independent from either linearly independent vectors in linear algebra, or forests in graph theory. So, to put it bluntly, what is this?

Matroids

In 1935, Hassler Whitney studied the properties that he noticed were shared by linear independence among vectors and acyclicity among edges. He demonstrated that a lot can be deduced from their shared properties alone, without needing to specifically reference vectors or edges at all (Whitney, 1935). He named the resulting object a *matroid*, a word that we can hope means to suggest that it relates in some way to a matrix.

A matroid consists of a finite set of elements, a *ground set*, together with a family of subsets denoted as *independent*, subject to three conditions both of our examples satisfy:

- the empty set is independent;

- every subset of an independent set is independent;
- if S and T are independent and S has fewer elements than T, then some element of T can be added to S keeping it independent.

A maximal independent set is called a *basis*, and a minimal dependent set is called a *circuit*.

With these definitions in hand, returning to our two examples, we can see that they are not merely both matroids, but are *the same** *matroid*. Indeed, we will say that two matroids are isomorphic if there exists a bijection between their ground sets that induces a bijection on their independent sets.

Anyone familiar with linear algebra is already well acquainted with the concept of linearly independent vectors. But perhaps less familiar, the graph version of a matroid also has a standard name. We can consider as our ground set the set of edges of a graph, and denote the forests of the graph as independent sets. Then, we can verify that this always forms a matroid which we (unimaginatively) call the graph's *cycle matroid*. Its bases are the spanning forests, and its circuits are the cycles.

So far, we have seen only one matroid. And, it arises both from a graph and from vectors. One might wonder if every graph's cycle matroid comes from vectors? Or, more generally, if every matroid does?

The incidence matrix

Whitney's first question, and ours, is which matroids actually come from vectors. For cycle matroids, the answer is concise: all of them.

The proof is not too difficult, and we present a brief version here. Given a graph G, denote one coordinate for each vertex and choose an orientation for each edge arbitrarily. Then define the column of an edge directed from u to v as $\mathbf{1}_v - \mathbf{1}_u$, writing $\mathbf{1}_w$ for the standard basis vector in the coordinate of vertex w. Together, these columns form the *incidence matrix* B, and the claim is that a set of columns is linearly independent exactly when the corresponding edges form a forest. This would show that the cycle matroid of G is *representable*: in general, we will say a matroid is representable whenever some collection of vectors realizes it in this way, that is, whenever it is isomorphic to a matroid coming from a set of vectors.

Now, suppose first that a set of edges contains a cycle. Traverse the cycle as $v_0, e_1, v_1, \ldots, e_k, v_k = v_0$ through distinct vertices $v_0, \ldots, v_{k-1}$. Each edge e_i joins v_{i-1} and v_i, however, the edges' fixed orientation may run either way along the traversal. Denote $\epsilon_i = +1$ if e_i is oriented from v_{i-1} to v_i, and $\epsilon_i = -1$ if it is not. In either case $\epsilon_i B_{e_i} = \mathbf{1}_{v_i} - \mathbf{1}_{v_{i-1}}$, so the signed sum around the cycle telescopes,

$$\sum_{i=1}^{k} \epsilon_i B_{e_i} = \sum_{i=1}^{k} \big(\mathbf{1}_{v_i} - \mathbf{1}_{v_{i-1}}\big) = \mathbf{1}_{v_k} - \mathbf{1}_{v_0} = \mathbf{0},$$

because the traversal ends at the same place it started. Thus, we have found a non-trivial linear combination of the columns of B summing to 0. Hence, any set of columns containing a cycle is dependent.

*up to isomorphism

In the other direction, suppose the edges form a forest and that some linear combination $\sum_e \lambda_e B_e = \mathbf{0}$. We must show that every coefficient λ_e is zero. This is left to the reader to verify.

Therefore, we have demonstrated that forests of a graph are exactly the independent sets of columns of its incidence matrix.

From our graph G a useful matrix to consider is B. The loop g has both ends at vertex 4, so its -1 and $+1$ fall in the same row and cancel. The parallel edges h and i join the same two vertices, so they are equivalent as column vectors. And, the bridge d is the only edge other than the loop meeting vertex 4, so its column is the only nonzero entry in row 4. Now, we see that we have collected a different set of vectors than the ones we began with, but that they represent the same matroid.

$$B = \begin{array}{cc} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \\ & \begin{matrix} a & b & c & d & e & f & g & h & i \end{matrix} \end{array}$$

Figure 3: The incidence matrix of G, in the same colour scheme as Figure **??**.

Two numbers are worth noting, since both will be mentioned again later. The rank of B, the largest number of independent columns, that is, the largest forest, is $n - c$ for a graph with n vertices and c components, a spanning tree when the graph is connected. For G this is $6 - 1 = 5$, the rank we found by hand. The dimension of the space of dependences among the columns is the first Betti number,

$$\beta = m - n + c,$$

which for G is $9 - 6 + 1 = 4$.

Do *all* matroids come from vectors?

Since we have now seen that every cycle matroid can be represented by vectors, and our only other example of a matroid thus far was already from linear algebra, it may be tempting to start thinking that every matroid can be formed from a set of vectors. In my opinion, this would be very disappointing: if every matroid came from vectors, then all of this would be nothing more than linear algebra, and our abstraction would have brought us nothing. We are lucky that it is not the case!

To build a counterexample, we first need one more way to picture a matroid. Recall that a basis is a maximal independent set. One of the consequences of the matroid axioms is that all bases have the same size, called the rank of the matroid. We'd say, for example, a rank-3 matroid is one where the largest independent sets have size 3.

For rank-3 matroids, we can draw their elements as points. A dependent triple is then drawn as three points on a line. A line in our drawing then just means, "these three elements form a circuit." Thus, we are allowed to draw points and lines first, and then consider after, if some actual set of vectors could have produced the same matroid. Such a picture of a matroid lets us choose which triples are lines, subject only to the matroid axioms. But geometry over a field is more restrictive than this.

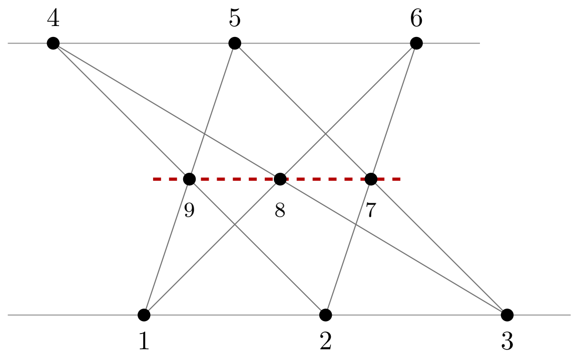


Figure 4: The Pappus configuration.

A relevant example is Pappus's theorem. Put three points $1, 2, 3$ on one line and three points $4, 5, 6$ on another. Now draw the six cross-lines joining them, and label the three opposite intersections by

$$9 = \overline{15} \cap \overline{24}, \qquad 8 = \overline{16} \cap \overline{34}, \qquad 7 = \overline{26} \cap \overline{35},$$

where $\overline{15}$ means the line through 1 and 5. Pappus's theorem says that $7, 8, 9$ must also be collinear. In other words, if the eight lines in Figure **??** come from points over a field, then the dashed ninth line is not optional.

After noticing all of this, the counterexample is simple. Keep the eight solid lines but not the ninth. We can define a matroid on the ground set $\{1, \ldots, 9\}$ as follows. The independent sets are all subsets of size at most 2, together with all triples except

$$\{1,2,3\},\ \{4,5,6\},\ \{1,5,9\},\ \{2,4,9\},\ \{1,6,8\},\ \{3,4,8\},\ \{2,6,7\},\ \{3,5,7\}.$$

Thus, the eight listed triples are the lines of the picture, while $\{7, 8, 9\}$ is independent. The reader can verify that this really is a matroid. Yet, it cannot be represented over any field. If vectors over a field represented it, then the eight listed triples would be collinear exactly as in the Pappus configuration. But Pappus's theorem would then force $7, 8, 9$ to be collinear too, contradicting the fact that our matroid declares $\{7, 8, 9\}$ independent.

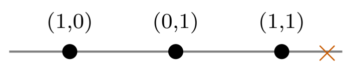
Non-Pappus is not the only matroid representable over no field. Another famous example is the Vámos matroid, whose obstruction is less easy to see from a picture.[†] In fact, examples like this are not rare at all. It is known that, as n grows, the fraction of matroids on n elements representable over any field tends to zero. So the surprising part is not that some matroids fail to come from linear algebra, but that the cycle matroid of G *is*.

[†] no, I will not be drawing it

Representable over *which* field?

Among the matroids that *are* representable, there is another question we have been ignoring: where do the entries of our vectors come from? Everything so far has secretly taken place over $\mathbb{R}$, but nothing in our definition of representability actually required this. We could just as well have asked for vectors with entries in $\mathbb{Q}$, or in the two-element field GF(2). As it turns out, a matroid representable over one field can fail to be representable over another.

Consider the matroid $U_{2,4}$, built by taking four elements and declaring a set independent exactly when it has at most two elements. To represent it, we would need four nonzero vectors in a plane, no two of them parallel. Over $\mathbb{R}$ this is no trouble at all. Over GF(2), however, we do not even get to start: the entire plane $\mathrm{GF}(2)^2$ contains only three nonzero vectors (Figure **??**), so there is no room for a fourth.



the three nonzero vectors of $\mathrm{GF}(2)^2$

Figure 5: Why $U_{2,4}$ is not representable over GF(2).

Another way a matroid can fail to be representable over a field has to do with the *characteristic* of the field. Consider the *Fano plane* F_7, a rank-3 matroid of seven elements whose picture has seven points and seven three-point lines. It can be shown that F_7 is representable over exactly the fields of characteristic 2. Perhaps more surprisingly, removing a single one of its lines produces the *non-Fano* matroid, which is representable over exactly the fields of characteristic *not* 2 (Oxley, 2011). Thus, we can organize matroids into a hierarchy, from worst to best (just kidding): representable over no fields, representable over some field, representable over only one characteristic, and representable over every field. Two of these classes have standard names. A matroid representable over GF(2) is called *binary*, and, since "the cycle matroid of some graph" is a bit wordy, a matroid of that kind is called *graphic*. This hierarchy is drawn in Figure **??**.

Regular matroids

We call a matrix *totally unimodular* if every square submatrix has determinant -1, 0, or $+1$. This definition may look arbitrary, but it is exactly the property that makes graphs special: any graphs incidence matrix B, is totally unimodular. We proceed with a proof sketch via induction. If some column of the submatrix is all zeros, the determinant is 0. If instead every column of the submatrix contains both its $+1$ and its -1, then the rows sum to the zero row, and the determinant is again 0. Otherwise, some column contains exactly one ±1, expanding the determinant along that column drops us to a smaller submatrix, where the induction hypothesis can be applied. Therefore, every square submatrix has determinant 0 or ±1.

Now, back to why we even care? Recall that a set of r columns, with r the rank of B, forms a basis exactly when its $r \times r$ submatrix has nonzero determinant. Total unimodularity strengthens "nonzero" to "±1." But ±1 is nonzero in every field, and 0 is zero in every field, so the very same

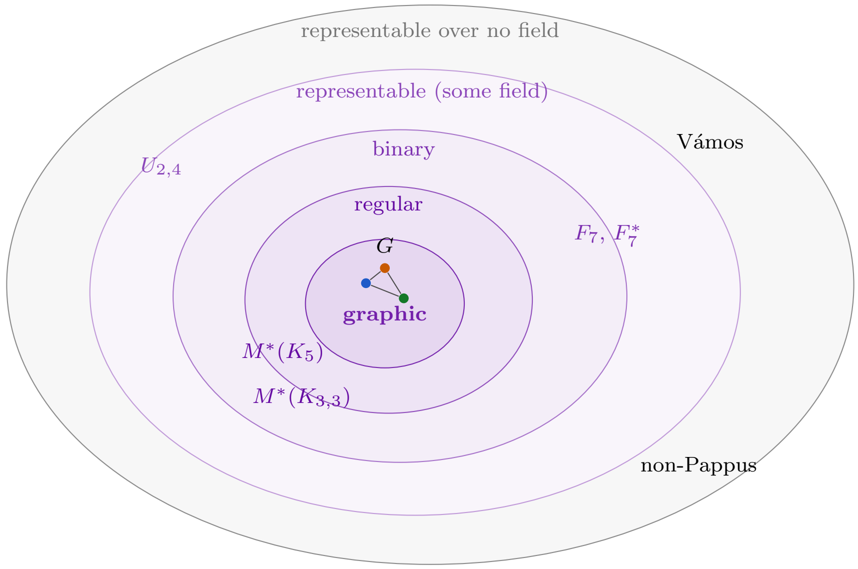


Figure 6: Matroids sorted by the fields they live over.

sets of columns come out as bases no matter which field we compute our determinants in. Such a matroid that is representable over every field is called *regular*, and we have just shown that every graphic matroid, is regular.

Tutte proved two fundamental theorems about regular matroids. The first, says that we do not actually need to check every field: a matroid is regular if and only if it is representable over both GF(2) and GF(3) (Tutte, 1958).

The second theorem requires one more definition. A *minor* of a graph is any graph obtained from it by deleting vertices/edges and *contracting* edges. It also makes sense to speak of one matroid containing another as a minor. With that language, Tutte's second theorem reads: a matroid is regular if and only if it contains none of $U_{2,4}$, F_7, and F_7^* as a minor (Tutte, 1958). Here F_7^* comes from the Fano plane by an operation called duality, which we will define properly soon. A list like this is called a *forbidden minor* characterization, and we will meet one more of them before the post is over.

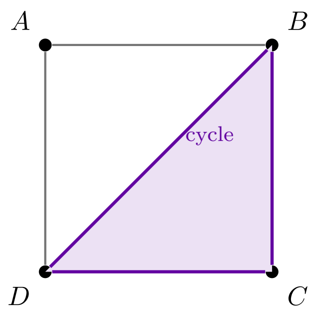
One might now wonder how much bigger than graphic the regular matroids really are. The answer, is: not much. Every regular matroid can be assembled from graphic matroids, their duals, and one 10-element matroid, glued together by a small set of operations (Seymour, 1980).

Duality and Planarity

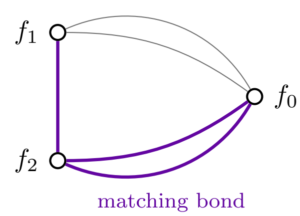
Shockingly, this all can be tied back into the planarity of a graph. To get there, let us consider $\mathrm{GF}(2)^m$, with one coordinate per edge. The matrix B determines two subspaces. The first is its null space, the *cycle space*: an edge set is sent to zero by B exactly when every vertex meets it an even number of times, which is the exact same as saying the edge set is a disjoint union of cycles. Its dimension is $\beta = m - n + c$, and for G that is 4, spanned by the loop, the two triangles, and the parallel pair. The second subspace is its row space, the *cut space*, whose members are the edge cuts; its dimension is $n - c$, which is 5 for G. The smallest nonzero members of the cycle space are the cycles, and the smallest nonzero members of the cut space are the *bonds*, the minimal cuts. Now, one can notice that these two subspaces are orthogonal complements of one another, as, a vector is orthogonal to every row of B exactly when B sends it to zero.

It can be shown similarly that a representable matroid is determined by the row space of any matrix representing it, and the *dual matroid* M^* is then defined by swapping that row space for its orthogonal complement. For the cycle matroid, this swap trades the cut space for the cycle space. So, for a graph, taking the matroid dual simply swaps cycles for bonds.

It helps to see this in a picture (Figure **??**). On the left there is a planar graph, and on the right its dual. The three edges of the shaded triangle form a cycle, and the three dual edges crossing them form a bond. One set of three edges, is a cycle on one side of the picture and an edge cut on the other.



(a) a planar graph, with a 3-cycle shaded



(b) its dual, with the matching bond

Figure 7: Duality interchanges cycles and cuts.

A graph is *planar* if it can be drawn in the plane with no edges crossing. G from earlier, is planar, as the drawing back in Figure **??** already has no crossings. Kuratowski's theorem says that: a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

For *any* graph G, the dual matroid $M(G)^*$ exists, since the orthogonal complement is always a matroid. The real question is whether that dual matroid happens to be the cycle matroid of an actual graph. Whitney settled this in 1932: $M(G)^*$ is graphic if and only if G is planar, and when G is planar, the graph realizing it is exactly the planar dual G^*, so $M(G)^* = M(G^*)$ (Whitney, 1932).

The interesting case is when G is *not* planar. $M(G)^*$ is still a perfectly good matroid, and it even has a name, *cographic*, but it is now the cycle matroid of no graph at all, nothing has those bonds as its cycles. The face-by-face dual construction only works when the orthogonal complement is allowed back into the world of graphs. So, being drawable in the plane is precisely the condition under which "take the orthogonal complement" keeps us among graphs.

Two lists of forbidden minors

We now have two forbidden-minor lists, and it turns out that they are secretly the same list. We saw the regular matroids characterized by three forbidden minors. The graphic matroids have a list of their own, again due to Tutte: a matroid is graphic if and only if it contains none of

$$U_{2,4}, \quad F_7, \quad F_7^*, \quad M^*(K_5), \quad M^*(K_{3,3})$$

as a minor (Tutte, 1959). The first three are exactly the forbidden minors for being regular, so the entire gap between regular and graphic comes down to the last two, $M^*(K_5)$ and $M^*(K_{3,3})$. That is, our rings nest,

$$\text{binary} \supset \text{regular} \supset \text{graphic}.$$

(a) K_5 (b) $K_{3,3}$

Figure 8: The two Kuratowski graphs. A graph is planar exactly when it contains neither as a minor, and their matroid duals $M^*(K_5), M^*(K_{3,3})$ are the two extra forbidden minors separating graphic matroids from regular ones.

That the two lists share these graphs is no coincidence, and it takes only one line to see why. Dualizing twice returns a matroid to itself, so $M^*(K_5)$ is graphic if and only if $M(K_5)$ is cographic, which by Whitney's theorem happens if and only if K_5 is planar, which it is not. The same argument applies to $K_{3,3}$. Because K_5 and $K_{3,3}$ (Figure **??**) are the *minimal* non-planar graphs, their duals are the minimal matroids that are cographic without being graphic. So, the graphs that obstruct planarity, after being passed through duality, become the matroids that obstruct being a graph at all.

So, what have we done?

We started with two questions that could hardly look less alike, and found that they had exactly the same answer. Whitney's matroid gave that shared answer a name. The incidence matrix showed that every graph really is a set of vectors, total unimodularity showed that graphs are representable in the strongest sense there is, and duality, of all things, decided which graphs can be drawn flat. So, linear algebra and graph theory were never as far apart as they seemed; the connection was simply never about eigenvectors. If you are still that extremely specific reader from the first paragraph, I hope you are satisfied. I know I am.